WORKING PAPERS

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RATE AND THE EXPECTED RATE OF INFLATION

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WORKING PAPER NO. 41

May 1980

BUREAU OF ECONOMICS
FEDERAL TRADE COMMISSION
WASHINGTON, DC 20580

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* I am grateful for comments from Thomas Cooley, James Langenfeld, Stephen LeRoy, and John Pippenger.

I. Introduction

The unprecedented inflation experienced by the United States since the mid 1970's clearly demonstrates the need to better understand the behavior of the real interest rate and its role in the economy. Unless the real interest rate is independent of the expected rate of inflation (which only few economists would expect in a disequilibrium context), the real rate may be significantly affected by periods of high inflation, with consequent effects on income flows into saving and investment. Unfortunately, little convincing empirical evidence exists concerning the temporal properties of the real interest rate or its relationship with other variables. This deficiency in the literature is largely related to the unobservability of the expected rate of inflation, and hence the real interest rate.

Despite the broad empirical support for the efficiency of financial markets,[1] this powerful framework has not been fully utilized to measure the real interest rate or its relationship to the expected rate of inflation. The present paper fills this gap in the literature by estimating models of the real interest rate and the expected inflation rate. The Kalman filter (an econometric procedure which can efficiently estimate imperfectly observed variables) is shown to be necessary for optimal estimates from these models. After the two series are estimated, the relationship between the real interest rate and the expected rate of inflation is estimated by standard regression analysis.

An examination of the relationship between these variables is important not only for its own sake, but also because a necessary assumption for much of the empirical work concerning the relationship between nominal interest rates and inflationary expectations is that the real interest rate is either constant or independent of the expected rate of inflation. This assumption has been commonly used even though very little empirical work has been done to assess its validity and several theoretical models suggest an inverse relationship between these variables. The econometric results reported here present very clear evidence that a strong inverse relationship does exist between the real interest rate and the expected rate of inflation.

Section II presents a brief review of past attempts to measure the real interest rate and characteristics of its temporal movements. Section III discusses the Kalman filter. Section IV presents estimates of the real interest rate and the expected rate of inflation, along with the estimated relationship between these variables. Finally, section V discusses the implications of the empirical results.

II. Measuring the Real Interest Rate

A. Ex-Post and Ex-Ante Real Interest Rates

The ex-ante real interest rate is the expected rate of return on a debt instrument in terms of commodities. Its relationship to the nominal interest rate is generally expressed as follows:[2]

$$\rho_t = R_t - \dot{P}^e_{t+n/t} \qquad (2.1)$$

where R_t is the nominal interest rate on an n period debt instrument, ρ_t is the ex-ante real interest rate on this security, and $\dot{P}^e_{t+n/t}$ is the expected rate of inflation between periods t and t + n based on the information available up to period t. Note that ρ is not observable because $\dot{P}^e_{t+n/t}$ is not observable.

The ex-post real interest rate on an n period debt instrument is the rate of return in terms of commodities which is actually obtained when the security is held to maturity. It is defined as follows:

$$y_t = R_t - \dot{P}_{t+n/t} \qquad (2.2)$$

where R_t is as previously defined and $\dot{P}_{t+n/t}$ is the actual inflation rate between periods t and t + n. Thus, the ex-post real interest rate is observable, but only at the end of the period over which it is defined. As a result, this interest rate does not affect the behavior of economic agents, since spending and portfolio decisions are based on the prospective real yield on securities when they are purchased.

The relationship between the ex-ante and ex-post real interest rates, which will be important in the empirical analysis, is derived by noting that

$$\dot{P}_{t+n/t} = \dot{P}^e_{t+n/t} + f_t$$

where f_t is the forecast error in inflationary expectations. Substituting this into equation (2.2) one obtains

$$
\begin{aligned}
y_t &= R_t - (\dot{P}^e_{t+n/t} + f_t) \\
&= \rho_t - f_t \qquad (2.3)
\end{aligned}
$$

This equation states that the ex-post real interest rate is equal to the ex-ante real interest rate minus the forecast error in the rate of inflation which is expected to prevail between periods t and t + n.

Since the present study concerns itself with a quarterly model of the real interest rate on three months Treasury Bills, a one period ahead forecast of inflation is appropriate. This insures that if this market is efficient in its inflationary expectations, f_t will be a white noise process.[3] This is true since, by Fama's definition, market efficiency means that the distribution of the market's expectation of inflation coincides with the true distribution of inflation. Thus, if the market is assumed to set its expected rate of inflation at the mean of its distribution, the expected rate of inflation will be an unbiased estimator of the actual rate of inflation which occurs over this period. This implies that the forecast errors in all periods have a mean of zero and are uncorrelated over time, which is to say that they are a white noise process.

B. Empirical Models of the Real Interest Rate

At one time many economists viewed the real interest rate as being either constant or orthogonal to the expected inflation rate. This belief was derived from Irving Fisher's full equilibrium, comparative statics model which assumed an absence of money illusion. When this view of real interest rate determination is combined with the assumption of autoregressive expectations, the following model results for a one period debt instrument:

$$R_t = \rho + \dot{P}^e_{t+1/t} \tag{2.4}$$

$$\dot{P}^e_{t+1/t} = \sum w_i \dot{P}_{t-i} + u_t \tag{2.5}$$

This model allows one to estimate a real interest rate series in two ways. First, the reduced form of equations (2.4) and (2.5) can be estimated with the constant term estimating the real rate. This method constrains the estimated real interest rate series to be constant. Alternatively, the real interest rate can be allowed to vary if one generates an inflationary expectations series from equation (2.5), which can be subtracted from the nominal interest rate to produce a real interest rate series. An approach very similar to this was used by the Federal Reserve Bank of St. Louis[4] to generate a real interest rate series.[5]

Although these approaches have the advantage of being easy to implement, they can be criticized on a number of grounds. The first approach, which might be called the extreme Fisherian model, is the most naive. Its basic premise that the real interest rate is constant is contrary to almost all macro theory, which views the real interest rate as an endogenous variable. While the second approach does not regard the real interest rate as a constant, it also has several serious faults. First, the dynamic relationship between interest rates and inflationary expectations under uncertainty is assumed to be identical to the comparative static relationship between them under certainty. Rutledge (1974, 1977) has shown that Fisher did not believe this to be true. In addition, the formation of inflationary expectations is likely to be more complex than the autoregressive formulation given in equation (2.5).

Pesando and Yatchew (1977) use a similar approach to the estimation of a real interest rate series, but differ in the method used to estimate the expected rate of inflation. This series is estimated by invoking the rational expectations hypothesis and assuming that expectations efficiently incorporated the information contained in past inflation and interest rates. A real interest rate series is then obtained by subtracting the inflationary expectations series from the nominal interest rate.

This method of constructing a real interest rate series appears to be an improvement on the "Fisherian" method since it uses a broader information set and constrains the inflationary expectations series to be rational; however, some limitations remain. First, it does not account for changes in inflationary expectations which result from sources besides the extrapolation of current and lagged rates of inflation and interest rates. The usefulness of other information sources in forecasting inflation is clear from recent experience with agricultural prices and the O.P.E.C. cartel. Second, the treatment of the real interest rate as a residual has a tendency to make movements in estimated real interest rates very similar to movements in nominal interest rates because of the smoothness of the inflationary expectations series. It is not clear that one would expect the real interest rate to be this volatile or its movements to mirror the movements in the nominal interest rate so closely. Furthermore, the smoothness of the inflationary expectations series appears to be inconsistent with market efficiency in forecasting inflation since the forecasting errors will tend to be autocorrelated. A final objection is that interest rates and inflation rates are highly autocorrelated, which implies a tendency for the polynomial distributed lags used by Pesando and Yatchew to show spurious relationships. These last two issues raise the question of whether the synthetic inflationary expectations series actually use the information set rationally.

Elliot (1977) takes a much different approach to measuring the ex-ante real interest rate. Several theoretical models of real interest rate determination are developed and empirical versions of these models are estimated. He then decides which model predicts the ex-post real interest rate with smallest root mean squared error based on in-sample and post-sample prediction tests.[6] This model is then used to estimate a synthetic series of ex-ante real interest rates which are the fitted values of the model.

Of particular interest is the method which Elliot uses to model the effects of various explanatory variables on the unobserved ex-ante real interest rate; the explanatory variables are regressed on the ex-post real interest rate. The use of the ex-post real rate as the dependent variable adds another error term to the equation if the true dependent variable is the ex-ante real rate. To see this more clearly, suppose one is interested in estimating the relationship between ρ_t and a given explanatory variable, X_t,

$$\rho_t = a + bX_t + e_t \tag{2.6}$$

but y_t is used in place of ρ_t. This implies the following model is estimated.

$$y_t = a + bX_t + e_t - f_t \tag{2.7}$$

If f_t is a zero mean white noise process and is determined independently of ρ_t and the explanatory variables (which will be true in an efficient market), then parameter estimates of (2.7) will be consistent and unbiased estimates of the parameters of (2.6) and the in-sample fitted series of real interest rates will be unbiased estimates of the true real rate since regression analysis constrains the expected value of the error term to be zero. Since the fitted values, $\hat{\rho}_t$, are obtained by setting the error term ($e_t - f_t$) to zero, this method will in principle filter the forecast error out of the series.

Elliot's method of estimating a real interest rate series offers the appealing qualities that real interest rates are calculated directly, and not as a residual, which should make their movements less erratic. Furthermore, this method makes use of ex-post inflation data which, given the recent evidence in favor of market efficiency, probably contain more information about the market's expected rate of inflation than do lagged inflation rates. However, the empirical framework used by Elliot to estimate the real interest rate series he reports is inefficient relative to the framework used in the present study because it does not efficiently incorporate the measurements (y_t) into the calculation of real interest rates. As will be shown, The Kalman filter optimally incorporates information from observations on y_t as well as information from a model of the real interest rate. The use of this additional information will significantly improve the quality of the estimated series.

III. The Kalman Filter

The Kalman filter is used here to estimate an unobservable time series variable which is measured with random error.[7] The true values of this variable (which is the ex-ante real interest rate in the case considered here) are called the state of the system. Maximum likelihood estimates of the state are obtained over a given sample period by optimally combining the imperfect measurements (the ex-post real rates) with estimates of the state obtained from a model. These estimates are obtained recursively, beginning with the first period of the sample and ending with the last. The model of the state is initially assumed to be known in the discussion below; however, after the filtering procedure is discussed a method of model estimation is presented.

The relationship between the unobservable state variable, x_t, and the observable measurement variable, y_t, is assumed to take the form

$$y_t = x_t + v_t \tag{3.1}$$

where $E(v_t) = 0$

v_t is independent of x_t

$$E(v_t v_s) = R \text{ for } s = t$$
$$= 0 \text{ otherwise}$$

In addition, the state is assumed to evolve by the process

$$x_t = \phi x_{t-1} + Dz_t + w_t \tag{3.2}$$

where $E(w_t) = 0$

$$E(w_t w_s) = Q \text{ for } t = s$$
$$= 0 \text{ otherwise}$$

z_t is a vector of exogenous variables

D is the coefficient vector of z_t

The error terms v_t and w_t are also assumed to be independent.

Equation (3.2) is used to estimate the state conditional on the value of the lagged state, the exogenous variables, and the model.

$$\bar{x}_{t/t-1} = \phi \bar{x}_{t-1/t-1} + Dz_t \tag{3.3}$$

This estimator of x_t is called $\bar{x}_{t/t-1}$ since it does not make use of the measurement y_t. It is an intermediate step in the construction of the estimate, $\bar{x}_{t/t}$, which optimally incorporates y_t. Note that since the true value of the state vector in period t - 1 is not known, its optimal estimator, $\bar{x}_{t-1/t-1}$, is used in its place in equation (3.3).

Under the assumption that the measurement error (y_t - x_t) and the structural error (x_t - $\bar{x}_{t/t-1}$) are independent and normally distributed with a mean of zero and variances of R and $M_{t/t-1}$ respectively, maximum likelihood estimates of x_t

are obtained by minimizing equation (3.4) below.[8]

$$F = \frac{(x_t - \bar{x}_{t/t-1})^2}{M_{t/t-1}} + \frac{(y_t - x_t)^2}{R} \tag{3.4}$$

The intuitive logic of minimizing equation (3.4) is clear. A weighted sum of measurement error squared and forecast error squared is minimized with the weights dependent on the variances of these two sources of error. Thus, the larger (smaller) is the variance of the structural error relative to the variance of the measurement error, the less (more) weight is assigned to structural error relative to measurement error and vice versa. Another way of viewing this is to say that the larger (smaller) is the variance of the structural error relative to the variance of the measurement error, the less (more) informative is $\bar{x}_{t/t-1}$ relative to y_t in constructing the optimal estimate $\bar{x}_{t/t}$. In the limit, if structural (measurement) error variances were infinite while measurement (structural) error variances were finite, no weight at all would be given to the former (latter) source of variation when determining $\bar{x}_{t/t}$.

One can solve for the maximum likelihood estimator of x_t by differentiating equation (3.4) with respect to x_t and setting the result equal to zero. Thus,

$$M^{-1}_{t/t-1}(x_t - \bar{x}_{t/t-1}) - R^{-1}(y_t - x_t) = 0$$

This implies that the maximum likelihood estimator of x_t, $\bar{x}_{t/t}$, will satisfy

$$(M^{-1}_{t/t-1} + R^{-1})\, \bar{x}_{t/t} = M^{-1}_{t/t-1}\, \bar{x}_{t/t-1} + R^{-1}y_t$$

Solving for $\bar{x}_{t/t}$ one obtains[9]

$$\bar{x}_{t/t} = \bar{x}_{t/t-1}+(M_{t/t-1})(R+M_{t/t-1})^{-1}(y_t-\bar{x}_{t/t-1}) \tag{3.5}$$

This equation also has a very intuitive interpretation. If no measurement error exists, R is zero and $\bar{x}_{t/t}$ becomes y_t. At the other extreme, if x_t is not measurable ($R = \infty$), $\bar{x}_{t/t}$ is simply $\bar{x}_{t/t-1}$. Equation (3.7) can also be written as

$$\bar{x}_{t/t} = \bar{x}_{t/t-1} + K_t e_t \tag{3.6}$$

where

$$e_t = y_t - \bar{x}_{t/t-1} \tag{3.7}$$

$$K_t = M_{t/t-1}(M_{t/t-1} + R)^{-1} \tag{3.8}$$

In this formulation the e_t are the residuals of the one step ahead forecast of the state equation, and K_t is called the Kalman gain or the optimal prediction correction factor.

Equations (3.3) and (3.6)-(3.8) are the basis of the filtering procedure. The process begins with equation (3.3), which defines the predicted value of x_t from the model. This allows the use of equations (3.6)-(3.8), which define $\bar{x}_{t/t}$, but the unknown variance, $M_{t/t-1}$, is introduced. Therefore, the next task is to derive an expression for $M_{t/t-1}$ in terms of known quantities.

Subtracting equation (3.3) from (3.2), one obtains

$$x_t - \bar{x}_{t/t-1} = \phi(x_{t-1} - \bar{x}_{t-1/t-1}) + w_t$$

Thus,

$$M_{t/t-1} = E(x_t - \bar{x}_{t/t-1})(x_t - \bar{x}_{t/t-1})$$

$$= E[\phi(x_{t-1} - \bar{x}_{t-1/t-1}) + w_t][\phi(x_{t-1} - \bar{x}_{t-1/t-1}) + w_t]$$

Since

$$M_{t-1/t-1} = E(x_{t-1} - \bar{x}_{t-1/t-1})(x_{t-1} - \bar{x}_{t-1/t-1})$$

the above equation for $M_{t/t-1}$ simplifies to

$$M_{t/t-1} = \phi^2 M_{t/t-1} + Q$$

which allows $M_{t/t-1}$ to be calculated.

The final step is the calculation of $M_{t/t}$, which after lengthy manipulations can be shown to be equal to[10]

$$M_{t/t} = (1 - K_t)M_{t/t-1}.$$

Given the values of $\bar{x}_{t/t}$ and $M_{t/t}$ just derived, the procedure can be repeated to estimated $\bar{x}_{t+1/t+1}$ and $M_{t+1/t+1}$. Estimates of the state and its variance can also be obtained recursively in all subsequent periods for which the measurement vector is available. Thus, when appropriate initial values of the state ($x_{o/o}$) and its variance ($M_{o/o}$) are determined, these variables can be estimated over the entire sample period by the filter.[11]

Starting values can be determined in several ways. First, Cooley, Rosenberg, and Wall (1977) suggest using the smoothed estimates of the state and its variance in the initial period. Another approach is applicable if the system (equation 3.3) is stationary. This involves setting the initial value of the state and its variance to their steady state values. For example, if the state equation is

$$x_t = \phi x_{t-1} + w_t$$

then x would have a steady state mean of zero and a variance of $var(w_t)/(1 - \phi^2)$.[12] This method was used in the present study primarily because of its simplicity.

To implement the model, its parameters--R, Q, ϕ, and D--must be estimated. Since y_t and $\bar{x}_{t/t-1}$ are normally distributed and unbiased estimators of x_t, the residuals (e_t) will be normally distributed with zero mean and finite variance V_t, where

$$V_t = M_{t/t-1} + R$$

Thus, the log-likelihood function of the parameter vector (θ), given e_t (t = 1, ... T), is

$$\ell(\theta/e_t) = -\frac{T}{2}\ell n 2\pi - \frac{1}{2}\sum_{t=1}^{T}(\ell n V_t + e_t^2 V_t^{-1})$$

The parameter vector which produces the set of residuals and V_t's which maximizes this expression is the maximum likelihood estimate of θ. Since the first term is a constant, minimization of the objective function below will produce identical parameter estimates.

$$J(\theta/e_t) = \sum_{t=1}^{T}(\ell n V_t + e_t^2 V_t^{-1}) \qquad (3.9)$$

After this function is concentrated with respect to the initial value of the state and its covariance matrix (by setting these variables at their steady state values) it is minimized by means of a Davidson-Fletcher-Powell optimization algorithm.[13] The parameter values associated with the minimum are then reported and used in the filtering procedure.

IV. Empirical Results

This section uses the Kalman filter to estimate the unobservable state of a system, which can be interpreted as the ex-ante real interest rate under the assumption that the market's forecast error for inflation is white noise. A series of inflationary expectations is then estimated in an analogous manner which allows the relationship between the real interest rate and the expected inflation rate to be analyzed. Quarterly data is used over the 1956-79 sample period.

A. Estimating a Real Interest Rate Series

The empirical analysis begins by reporting an estimated model of the form:

$$y_t = \rho_t + f_t \tag{4.1}$$

$$\rho_t = \phi \rho_{t-1} + g_t \tag{4.2}$$

The ex-post real interest rate (y_t) is calculated as the nominal interest rate on three month Treasury Bills[14] (measured on the last day of each quarter) minus the actual inflation rate over the subsequent quarter. The error terms are assumed to have the properties discussed in section III. The state is interpreted as the ex-ante real interest rate since the ex-post real interest rate can be expressed as the sum of the ex-ante real rate and the forecast error in inflationary expectations (see equation 2.3). This produces an equation the form of (4.1) if the forecast error is white noise.

$$y_t = \rho_t + f_t \qquad \text{VAR}(f_t) = \underset{(.406)}{1.742} \tag{4.3}$$

$$\rho_t = \underset{(.068)}{.912}\, \rho_{t-1} + g_t \qquad \text{VAR}(g_t) = \underset{(.177)}{0.345}$$

The estimated model is displayed above. It was deemed to be an appropriate representation of the data since a series of diagnostic tests did not even hint at any defects.[15] This model indicates that the expected real rate of return on Treasury Bills follows a stationary first order autoregressive process.[16] It should be noted, however, that the model is consistent with the hypothesis that the real rate follows a random walk, since ϕ is within two standard deviations of unity. A final implication of the model is that errors in inflationary expectations (f_t) are shown to be the dominant cause of variation in the ex-post real interest rate, since var(f_t) exceeds var(g_t). This finding is similar to the results reported by Nelson and Schwert (1977).

The estimated values of the state, $\bar{\rho}_{t/t}$, which result from applying the filtering algorithm to this model are displayed in the first column of Table IV.1. These values are simply weighted averages of the two imperfect measures of the ex-ante real rate: the ex-post real rate and the predictions of the model (equation 4.2). The weights are determined by the informativeness of each source.

The most notable features of the estimated series are the large amount of variation in the estimated real interest rates and the fact that they are negative over the periods 1956-58 and

1973-79. This last fact would be disturbing if the interest rate considered here were a long-term rate, since presumably no one would lend money if a higher rate of return could be earned by simply holding real assets. However, three month Treasury Bills serve as substitutes for money as well as for real assets. Thus, even when the expected rate of inflation exceeds the nominal interest rate on Treasury Bills, their safety, liquidity, and greater return than money apparently make them attractive to investors. One can also rationalize expected negative real returns in the Treasury Bill market by transactions costs and preferred habitats. Thus, even if investors expect the rate of inflation over the next three months to exceed the current Treasury Bill rate, the costs of finding and purchasing the appropriate real assets may exceed the gains of their ownership in the short run.

One can also make a very heuristic argument that market efficiency implies negative real rates over the above period, based on the fact that ex-post real rates were consistently negative. If market participants did not expect a negative real interest rate over these periods, inflationary expectations were consistently biased downwards, which implies that markets were not efficient.

B. Estimating the Relationship Between the Real Interest Rate and The Expected Rate of Inflation

This section begins with the estimation of a series of expected inflation rates, which are estimated in a manner very similar to the procedure used to estimate the real interest rate

TABLE IV.1

ESTIMATED VALUES OF THE REAL INTEREST RATE ($\bar{\rho}_{t/t}$) AND THE EXPECTED RATE OF INFLATION ($\bar{\delta}_{t/t}$)

		$\bar{\rho}_{t/t}$	$\bar{\delta}_{t/t}$	R_t	$(R_t - \bar{\delta}_{t/t})$
1956	I	-.339	1.051	2.30	1.249
	II	-1.187	4.655	2.65	-2.005
	III	-.938	3.590	2.31	-1.280
	IV	-.573	2.755	2.65	-.105
1957	I	-.574	3.156	3.08	-.076
	II	-.764	3.944	3.03	-.914
	III	.104	2.690	3.37	.680
	IV	-.143	3.217	3.59	.373
1958	I	-.520	3.504	1.60	-1.904
	II	-.114	2.563	1.22	-1.343
	III	.108	1.216	.57	-.646
	IV	.912	.837	2.55	1.713
1959	I	1.027	.617	2.75	2.133
	II	.684	1.706	2.87	1.164
	III	.801	1.785	2.99	1.205
	IV	1.765	.901	4.14	3.239
1960	I	1.594	1.584	4.00	2.416
	II	1.610	1.442	3.03	1.588
	III	.935	1.754	2.10	.346
	IV	1.224	1.144	2.25	1.106
1961	I	1.469	.821	2.35	1.529
	II	.985	1.275	2.27	.995
	III	1.156	1.054	2.25	1.196
	IV	1.040	1.379	2.69	1.311
1962	I	1.040	1.379	2.69	1.311
	II	.946	1.270	2.74	1.470
	III	1.293	1.522	2.87	1.348
	IV	1.390	.880	2.72	1.840
1963	I	1.521	.821	2.94	2.119
	II	.942	1.892	2.90	1.008
	III	1.258	1.248	3.25	2.002
	IV	1.568	1.497	3.49	1.993

TABLE IV.1 (continued)

		$\bar{\rho}_{t/t}$	$\bar{\delta}_{t/t}$	R_t	$(R_t - \bar{\delta}_{t/t})$
1964	I	1.848	1.019	3.50	2.481
	II	1.740	1.360	3.46	2.100
	III	1.748	1.092	3.47	2.378
	IV	1.908	1.244	3.55	2.306
1965	I	1.550	1.329	3.88	2.551
	II	1.634	2.145	3.90	1.755
	III	1.602	1.541	3.82	2.279
	IV	1.293	1.795	4.07	2.275
1966	I	.962	3.315	4.64	1.325
	II	.863	3.104	4.65	1.546
	III	1.061	3.571	4.75	1.179
	IV	.206	4.956	5.23	.274
1967	I	1.816	3.736	4.50	.764
	II	.988	3.906	3.75	-.156
	III	.823	3.694	4.12	.426
	IV	.654	3.703	4.58	.877
1968	I	.491	3.981	4.84	.859
	II	.433	4.550	5.48	.930
	III	.300	4.566	4.97	.404
	IV	.410	4.276	5.46	1.184
1969	I	-.324	5.619	6.19	.571
	II	.019	5.651	5.92	.269
	III	.418	5.367	7.00	1.633
	IV	.512	5.707	6.97	1.263
1970	I	.678	6.139	7.72	1.581
	II	1.139	5.742	6.84	1.098
	III	1.037	5.337	6.40	1.063
	IV	1.473	4.645	5.69	1.045
1971	I	.759	4.089	4.12	.031
	II	.414	4.606	3.84	-.766
	III	1.212	3.482	5.24	1.758
	IV	1.092	3.103	4.26	1.157
1972	I	.722	3.288	3.40	.112
	II	.434	3.518	3.62	.102
	III	.312	3.500	3.77	.270
	IV	-.103	3.478	4.75	1.272

TABLE IV.1 (continued)

		$\bar{\rho}_{t/t}$	$\bar{\delta}_{t/t}$	R_t	$(R_t - \bar{\delta}_{t/t})$
1973	I	-1.114	6.029	5.76	-.269
	II	-1.586	6.071	6.29	.219
	III	-.405	8.580	8.38	-.200
	IV	-2.627	8.838	7.48	-1.358
1974	I	-3.057	10.303	7.31	-2.993
	II	-3.175	10.866	8.77	-2.096
	III	-3.312	12.149	7.94	-4.209
	IV	-1.969	10.473	7.78	-2.693
1975	I	-1.477	8.764	5.66	-3.104
	II	-1.964	9.056	5.50	-3.556
	III	-1.286	7.626	6.35	-1.276
	IV	-.285	6.556	5.57	-.986
1976	I	-.256	5.297	4.80	-.497
	II	-.682	5.984	4.88	-1.104
	III	-.255	5.624	5.11	-.514
	IV	-.156	5.193	4.84	-.353
1977	I	-1.811	7.252	4.74	-2.512
	II	-1.793	7.022	4.75	-2.272
	III	-.847	5.807	5.35	-.457
	IV	-.453	5.828	6.22	.392
1978	I	-1.259	7.298	6.41	-.888
	II	-2.361	8.891	6.31	-2.581
	III	-2.086	8.784	6.88	-1.904
	IV	-.966	8.292	8.85	.558
1979	I	-2.087	10.560	9.25	-1.310
	II	-2.958	12.198	9.50	-2.698
	III	-2.875	12.182	9.16	-3.022
	IV	-2.590	13.065	12.06	-1.005

series. As with the real interest rate model, the crucial assumption is market efficiency, which implies that the forecast error in inflationary expectations is a white noise process. Filtering models of the form

$$\dot{P}_{t+1} = \delta_t + f_t \tag{4.4}$$

$$\delta_t = \phi\delta_{t-1} + h_t \tag{4.5}$$

are estimated, where δ_t is the rate of inflation the market expects to prevail over period t + 1 based on information available through period t, and where f_t is the forecast error in inflationary expectations. Note that f_t in equation (4.4) is the negative of f_t in equation (4.1). Thus, if the estimated models of this chapter are meaningful, the two different estimates of the variance of f_t should be approximately equal. The estimated models which follow show this to be true.

The same diagnostic tests described in the previous section are applied to this model and it is found to be adequate. The final model used in the filtering is given below.

$$\begin{aligned} \dot{P}_{t+1} &= \delta_t + f_t & \quad \mathrm{var}(f_t) &= \underset{(.511)}{2.244} \\ \delta_t &= \underset{(.026)}{.985}\,\delta_{t-1} + h_t & \quad \mathrm{var}(h_t) &= \underset{(.463)}{1.050} \end{aligned} \tag{4.6}$$

The filtered values from this model are displayed in Table IV.1 under the column labeled $\bar{\delta}_{t/t}$.

Before these filtered values are used to examine the relationship between the real interest rate and the expected rate of inflation, the series is used to construct an alternative measurement of the real interest rate, which is obtained by subtracting the expected rate of inflation ($\bar{\delta}_{t/t}$) from the nominal interest rate (R_t). This series is displayed in the last column of Table IV.1. If the approach used in this chapter produces meaningful results, the two series, $\bar{\rho}_{t/t}$ and ($R_t - \bar{\delta}_{t/t}$), should be closely related. A comparison of columns 1 and 4 of Table IV.1 reveals that although the two series occasionally diverge noticeably, their movements are very similar, which is indicated by a correlation coefficient of .894. One obvious characteristic of the ($R_t - \bar{\delta}_{t/t}$) series is that it has a larger variance than the $\bar{\rho}_{t/t}$ series. Apparently the filtering smooths the $\bar{\delta}_{t/t}$ series somewhat so that when the real rate is calculated as a residual it becomes more volatile than when it is calculated directly. However, since these two distinct measures of the real interest rate show the same basic pattern of movement over time, this test of the modeling procedure does not appear to reveal any difficulties.

The relationship between the real interest rate and the expected rate of inflation was estimated as follows.

$$\bar{\rho}_{t/t} = \underset{(.156)}{.853} - \underset{(.032)}{.186}\bar{\delta}_{t/t} + \underset{(.073)}{.555}\bar{\rho}_{t-1/t-1}$$

A lagged dependent variable was used in order to fully account for the autoregressive structure of $\bar{\rho}_{t/t}$, and thus not induce a spurious relationship.[17] This provides convincing evidence in favor of the proposition that the real interest rate is inversely related to the expected rate of inflation. One interpretation of this result is that in a steady state with no inflation the real interest rate is about 0.85. However, as the expected rate of inflation increases, the cost of holding one's liquid assets in cash also increases. Thus, investors are willing to accept a progressively lower real yield on Treasury Bills to avoid the large losses in purchasing power associated with holding money.

V. CONCLUSIONS

This paper has estimated a series of real interest rates and expected inflation rates based on the assumption that inflationary expectations are efficiently formed. The most notable conclusions derived from the estimated series are that real interest rates showed considerable variation and were negative over 1973-79, and that a strong inverse relationship between real interest rates and expected inflation rates was shown to exist. These conclusions are important in themselves, since they imply that the real interest rate--an important macroeconomic variable--will be lower (and possibly negative) in periods of high inflation than in times of low inflation.

The finding of a significant inverse relationship between the real interest rate and the expected rate of inflation also casts a shadow on much of the empirical work concerning interest rates and inflationary expectations which assumes that the real rate is constant or orthogonal to the expected inflation rate. One can easily show that this biases downward the estimated impact of inflationary expectations on interest rates in models which assume expectations to be formed autoregressively (i.e., regressions of interest rates on current and lagged inflation rates). This finding also implies that Fama's (1975) test of market efficiency is not correct. While this conclusion is nothing new,[18] the present paper provides additional, and perhaps more clear evidence that the real rate is not constant and is dependent on the expected rate of inflation.

The finding that real rates remained negative over such an extended period of time is also an important conclusion of this paper. This is apparently the result of close substitutability between cash and Treasury Bills in portfolio management. Apparently, the safety and liquidity of Treasury Bills make investors willing to hold them at negative real yields, especially during periods of high inflation, when the opportunity cost of holding money is highest.

FOOTNOTES

1. Empirical evidence for market efficiency is contained in the following: Fama (1970), Fama (1975), Nelson and Schwert (1977), Phillips and Pippenger (1976).

2. To be precise this relationship is

$$\rho_t = \frac{R_t - \dot{P}^e_{t+n/t}}{1 + \dot{P}^e_{t+n/t}}$$

However, the present study will make the simplifying assumption usually made in the empirical literature that equation (2.1) describes this relationship. One can easily see that when inflation rates are moderate, little distortion is produced by the use of equation (2.1).

3. If a multiperiod forecast is appropriate, f_t will not be white noise. For example, in a two period ahead forecast, the forecast error,

$$f_t = \dot{P}_{t+2/t} - \dot{P}^e_{t+2/t}$$

is clearly a function of the inflation innovations: $\epsilon_{P,t+1}$ and $\epsilon_{P,t+2}$. Similarly, f_{t+1} is a function of $\epsilon_{P,t+2}$ and $\epsilon_{P,t+3}$ which implies a correlation between f_t and f_{t+1} because of their common association with $\epsilon_{p,t+2}$. Similar arguments can be made for longer term forecast spans.

4. See Yohe and Karnosky (1969).

5. Another variant of this model can be used to examine the impact of a variable (or set of variables) on the real interest rate.

$$R_t = \rho_t + \dot{P}^e_{t+n/t} + u_t$$

$$\dot{P}^e_{t+n/t} = \sum w_i P_{t-i}$$

$$\rho_t = a + \sum v_i X_{t-i}$$

This model has the following reduced form:

$$R_t = a + \sum v_i X_{t-i} + \sum w_i \dot{P}_{t-i} + u_t$$

The v_i coefficients can then be interpreted as the impact of X on the real interest rate under the assumption that current and lagged inflation rates affect the expected rate of inflation (and not the real rate of interest) and that the variable X affects only the real interest rate (and not the expected rate of inflation). This approach is not used here for several reasons. First, the present study seeks to determine the impact of inflation on the real interest rate and the above model cannot identify this relationship. Second, not many variables which strongly affect the real interest rate appear to be independent of the expected rate of inflation as is required by this model. Finally, it is doubtful that inflationary expectations can be meaningfully represented as an autoregressive process.

6. Ideally, these models of the ex-ante real interest rate would be rated on their ability to predict that variable rather than the ex-post real interest rate. But since the ex-ante real rate is unobservable, the forecasting test in terms of the ex-post real interest rate is used to approximate that standard.

7. Principal sources are: Athans (1974), Bryson and Ho (1969), Chow (1976), Cooley (1977), and LeRoy and Waud (1975).

8. The log-likelihood function of x_t can be written as follows (see Theil, p. 70):

$$\begin{aligned} &\ell(x_t/\bar{x}_{t/t-1}, y_t, M_{t/t-1}, R) \\ &= - \ln 2\pi - \tfrac{1}{2}\ln R - \tfrac{1}{2}\ln M_{t/t-1} \\ &\qquad - \tfrac{1}{2} \{(y_t - x_t)^2/R + (x_t - \bar{x}_{t/t-1})^2/M_{t/t-1}\} \end{aligned}$$

Since x_t only appears in the last term, maximization of the likelihood function with respect to x_t is identical to minimization of equation (3.4).

9. See LeRoy and Waud (1975), p. 9.

10. See LeRoy and Waud (1975), p. 11.

11. The filtered estimates, $\bar{x}_{t/t}$, efficiently use the information contained in the sample from the initial period to t. A further refinement can be made through smoothing in which the estimates of the state would incorporate all information in the sample (see Cooley, Rosenberg, and Wall, 1975). Smoothing was performed; however, the smoothed values did not pass a diagnostic test described in section IV.B in which an alternative real interest rate series was constructed. Thus, they are not reported.

12. The population mean is zero since this equation can be written in the form

$$x_t = \sum_{i=0}^{\infty} \phi^i w_{t-i}$$

where $E(w_{t-i}) = 0$. The population variance is equal to

$$\sigma^2 = E(x_t^2) = E(\sum_{i=0}^{\infty} \phi^i w_{t-i})^2$$
$$= Q(1 + \phi^2 + \phi^4 + . . .) = Q/(1-\phi^2)$$

13. Given an arbitrary set of starting values for the parameter vector, θ_0, the DFP algorithm iteratively modifies the parameter vector until one is obtained which minimizes equation (3.13). Each iteration begins with the parameter values estimated in the previous iteration (with the starting values used in the first iteration). Using these parameter values and initializing $x_{0/0}$ and $M_{0/0}$ as previously explained, the filtering equations produce a series of $\bar{x}_{t/t-1}$ and $M_{t/t-1}$ terms which are used to estimate e_t and V_t over all time periods. These latter terms are then used to evaluate the likelihood function and determine if a maximum has been reached. When the maximum likelihood estimates are obtained, the filtering is done one final time to obtain the set of filtered values, $\bar{x}_{t/t}$ and their covariance matrices, $M_{t/t}$, which are reported.

14. The three month Treasury Bill rate series was obtained from Federal Reserve Publication G-14 and from the records of the Federal Reserve Bank of St. Louis.

There are two reasons why the Treasury Bill rate was used here instead of an alternative short-term rate such as the Commercial Paper rate. First, the Treasury Bill rate has been almost universally used to measure the nominal interest rate in the literature on the determination of real interest rates and the relation between nominal interest rates and the expected rate of inflation. Thus, the use of this rate facilitates a comparison of this paper with the literature. Second, a 90-day Commercial Paper rate which was sampled on the last day of each quarter was not available.

15. First, since the innovations (e_t) of each model are assumed to be a white noise process in the derivation of the estimated parameters, the autocorrelation function of the estimated innovation series for each model was examined for evidence that this series is not white noise. Since none of the autocorrelation coefficients are larger than twice their estimated standard deviation in absolute value, the innovation series appear to be white noise in all models.

Second, the above model was tested for structural change by bisecting the sample period and estimating a model over each half. No evidence of change was present.

Other diagnostic tests added additional structure to the above models. The first such expansion made equation(4.2) into a second order autoregresssive process. This is accomplished by specifying the state equation as the two component vector process

$$\begin{bmatrix} \rho_t \\ \rho_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \rho_{t-1} \\ \rho_{t-2} \end{bmatrix} + \begin{bmatrix} g_t \\ 0 \end{bmatrix}$$

Estimates of ϕ_2 were not significantly different from zero in any of the models. As a result the original specification of a first order process appears to be correct.

An attempt was also made to incorporate a constant term into the state equation and estimate it in the form

$$\rho_t = k + \phi\rho_{t-1} + v_t$$

However, k was either insignificant or highly correlated with other parameters. Thus, this specification did not add to the informativeness of the model.

16. The autoregressive coefficient was not constrained to stationary values in the program used here.

17. See Granger and Newbold (1974, 1977).

18. See the American Economic Review (June 1977, pp. 469-496) for a series of critiques of Fama's (1975) paper. A rejoinder by Fama is also present.

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