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EQUILIBRIA ACHIEVED WITH COMMUNICATION

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Up to this point the effects of communication on the equilibrium behavior of the decisionmakers in a game have not been analyzed with the rigor expected of most game theoretic results. Nevertheless, the supposed effects are incorporated into the most fundamental solution concepts used for both the noncoopertive game and the cooperative game. In each case a story is told describing the effect of communication, and this story is used as the primary justification for the equilibrium concept used. In these stories we imagine an environment in which an agreement is made between the decisionmakers in some pre-game negotiation process and then each decisionmaker chooses a strategy which is actually played, a strategy which may or may not satisfy the agreement. In addition, before any choices are made this imagined environment specifies the sanctions which are to be imposed for violating any agreements, sanctions which determine the incentives for satisfying any agreements. Given such an environment, the stories told are used to demonstrate the reasonableness of using proposed solution concepts, which specify what agreements are likely to be made in such an environment.

More specifically, a noncooperative game is by definition a game which has no sanctions for violating any agreement. Any agreements are non-binding. In this case, a story is told that any equilibrium selection of strategies is achievable with a self-enforcing agreement, or in other words, it must be a Nash equilibrium. We expect that no agreement will be made in which some decisionmaker has an incentive to change his strategy unilaterally. With a cooperative game, by definition, there are implicit sanctions so severe for violating an agreement that no player ever chooses to do so. All agreements are binding. In this case, one story is told that in an equilibrium no agreement will be made in which any individual can improve upon its outcome regardless of the strategies chosen by others. This story leads to the concept of individual rationality. Another story is told that in an equilibrium no agreement will be made in which any coalition can improve upon its outcome regardless of the strategies chosen by others. This story leads to the concept of the core. Our intent here is to examine the conjecture that these stories can be supported by explicitly including a pre-game negotiation process and the applicable sanctions for violating any agreements, and generating the proposed solution concepts from the Nash equilibria and strong Nash equilibria of this game with communication. Our approach is consistent with the one suggested in Nash (1951) on tying the analysis of noncooperative and cooperative games together. Thus, we will construct this game with communication, examine its Nash equilibria and strong Nash equilibria, and determine the relationship between the equilibrium outcomes of this game and those proposed for the elemental game where communication is only implicit.

THE GAME WITH COMMUNICATION

Consider an elemental game given in strategic form. Let I be the index set of players, S^i be the set of feasible strategies for each player $i \in I$ and $\pi^i(s)$ be the payoff for player $i \in I$ given the selection of strategies $s \in S$.[1] The game with communication, constructed from the elemental game, has the same players as the elemental game, has sets of feasible strategies which are modified to incorporate a negotiation process, and has payoff functions which are modified to incorporate the sanctions imposed by the rules of the game for violating an agreement.

The strategies in the game with communication augment the actions taken in the negotiation process to the strategies in the elemental game. In the negotiation process the players in the game communicate by sending messages to each other. Among these messages are ones which are capable of indicating whether an agreement has been made between the players, and any agreement must specify the allowable strategies for each player to satisfy the agreement. While we may describe these allowable strategies for each player as a set, for the payoffs from an agreement to be well-defined we may without loss of generality indicate only one allowable strategy for each player in any agreement. This means an agreement is fully described by a selection of strategies $s \epsilon S$, which describes the actions each player must take to fulfill the agreement. This means that the messages sent in any negotiation process must contain such an s if an agreement is to be made. As we will see with our version of the noncooperative game, any part of the communication process which is non-binding does not affect equilibrium behavior, so that for describing equilibrium behavior all that is essential in the negotiation process is knowing what, if any, agreements are made. Given this, we will formalize a message from each decisionmaker as a choice of a selection of strategies, and an agreement is made if all players choose the same selection of strategies. After this negotiation process, where a message is sent by each player and these messages are known to all players, each player chooses a strategy from the elemental game. Thus, a strategy in the game with communication consists of a message and a choice of a strategy from the elemental game which may depend upon all messages which have been sent. This means for player $i \varepsilon I$ a strategy in the game with communication is a $\sigma^i = (m^i, f^i)$ where the message $m^i \varepsilon S$ and the choice of an elemental strategy is given by the function $f^i: \underset{i \varepsilon I}{X} S \rightarrow S^i$.[2]

The payoffs for the game with communication depend upon the sanctions imposed on the players for violating an agreement. If the player honors an agreement or if there is no agreement, the player receives just the payoff associated with the selection of strategies which is actually played from the elemental game. If the player violates an agreement, he receives this payoff minus a penalty, which describes the effect of the sanctions imposed upon him. Formally, the payoff for player $i \varepsilon I$ in a game with communication given a selection of strategies $\sigma = (m, f)$ is

$$\Pi^i(\sigma) = \begin{cases} \pi^i(f(m)) & \text{if } (\exists s \forall j) m^j = s \text{ and } s^i = f^i(m), \\ & \text{or } (\exists j, k)\ m^j \neq m^k \\ \pi^i(f(m)) - p^i(\sigma) & \text{if } (\exists s \forall j) m^j = s \text{ and } s^i \neq f^i(m) \end{cases}$$

where $p^i(\sigma)$ is the penalty imposed on i when an agreement is violated. Specifically, we are interested in the games with communication that correspond to the noncooperative game and the cooperative game. We will compare a noncooperative elemental game to the corresponding game with communication and no binding agreements, where $p^i(\sigma) \equiv 0$ so that $\pi^i(\sigma) = \pi^i(f(m))$. We will compare a cooperative elemental game to the corresponding game with communication and binding agreements, where the sanctions imposed yield the worst possible payoff so that

$$\Pi^i(\sigma) = \begin{cases} \pi^i(f(m)) & \text{if } (\exists s \forall j) m^j = s \text{ and } s^i = f^i(m), \\ & \quad \text{or } (\exists j,k)\ m^j \neq m^k \\ -\infty & \text{if } (\exists s \forall j)\ m^j = s \text{ and } s^i \neq f^i(m) \end{cases}$$

Now we wish to examine the Nash equilibria and the strong Nash equilibria of these games with communication and determine the relationship between the outcomes associated with these equilibria and those suggested as likely outcomes for the elemental games. A Nash equilibrium is a selection of strategies σ where there is no player i with a strategy $\bar{\sigma}^i$ such that $\Pi^i(\sigma/\bar{\sigma}^i) > \Pi^i(\sigma)$.[3] A strong Nash equilibrium is a selection of strategies σ where there is no coalition C with strategies $\bar{\sigma}^C$ such that $\Pi^i(\sigma/\bar{\sigma}^C) > \Pi^i(\sigma)$ for all $i \in C$. The set of (strong) equilibrium outcomes is $\{\Pi(\sigma)$: σ is a (strong) Nash equilibrium$\}$.

EQUILIBRIUM OUTCOMES IN GAMES WITH COMMUNICATION VS. THE ELEMENTAL GAME

First, we will compare the equilibrium outcomes of the game with communication and no binding agreements with those of the noncooperative elemental game. The following theorem implies that for a noncooperative game a negotiation process is not essential for determining the outcomes which occur in equilibrium, demonstrating that the same outcomes occur in equilibrium with or without a negotiation process. Equilibrium behavior is not affected by any non-binding communication. This means the game with communication and no binding agreements is essentially equivalent to the noncooperative elemental game, supporting the story told about the effects of a negotiation process on a noncooperative game. Also, this result justifies our use of messages which only include the potentially binding communication.

Theorem 1: The set of (strong) equilibrium outcomes in the game with communication and no binding agreements is identical to the set of (strong) equilibrium outcomes in the noncooperative elemental game.[4]

Now, we will compare the equilibrium outcomes of a game with communication and binding agreements to those outcomes proposed as most likely for a cooperative elemental game, those that are individually rational and those in the core. A complication arises in defining these two equilibrium concepts for a cooperative game, as there are two ways to define when a coalition (or individual) can "improve upon" an outcome.[5] We will say a coalition C can alpha-improve upon $\pi(s)$ if there exists an $\bar{\bar{s}}^C \epsilon S^C$ for all $\bar{s} \epsilon S$ such that $\pi^i(\bar{s}/\bar{\bar{s}}^C) > \pi^i(s)$ for all $i \epsilon C$, and a oalition C can beta-improve upon $\pi(s)$ if for all $\bar{s} \epsilon S$ there exists $\bar{\bar{s}}^C \epsilon S^C$ such that $\pi^i(\bar{s}/\bar{\bar{s}}^C) > \pi^i(s)$ for all $i \epsilon C$. Note that the only difference between the two definitions is the order of the qualifiers. The alpha definition stresses what a coalition can guarantee for itself, while the beta definition stresses what a coalition cannot be prevented from achieving. It is as if for an alpha-improvement the coalition C announces its strategies first with the opposing coalition responding as it wishes, and for a beta-improvement the opposing coalition announces its strategies first with C responding.

We use these definitions of "improving upon" in our definitions of individual rationality and the core. An outcome is individually rational if no individual can improve upon it, and an outcome is in the core if no coalition can improve upon it. Note that, if a coalition can alpha-improve upon an outcome it can always beta-improve upon the outcome, so that the set of beta-individually rational outcomes is a subset of the set of alpha-individually rational outcomes and the beta-core is a subset of the alpha-core. Also, even while we are given a pair of definitions for each concept, the literature does not provide a strong justification for using either one of these definitions over the other.[6] Relationships between these solution concepts for the cooperative game and both the Nash and strong Nash equilibrium outcomes from the associated games with communication and binding agreements are given in the theorems that follow.

Theorem 2: The set of equilibrium outcomes for the game with communication and binding agreements is identical to the set of beta-individually rational outcomes for the elemental game.

The interesting part of this theorem is that the set of equilibrium outcomes is big enough to be equal to the set of beta-individually rational outcomes. We see directly from the definitions that any equilibrium from any game with communication must be beta-individually rational. This includes the game with communication and no binding agreements, a game essentially equivalent to the noncooperative elemental game. The effect of adding penalties is to decrease the incentive to change strategies once an agreement is made, so that the set of equilibrium outcomes may have some new members. The interest comes from knowing that the severe penalties used in the game with communication and binding agreements guarantee there are enough new members in the set of equilibrium outcomes so that it must equal the set of beta-individually rational outcomes.

Here we show the structure of the game with communication and binding agreements provides the same strategic opportunities as those described by the story justifying the use of beta-individually rational outcomes. Since added penalties only decrease the incentive to violate any agreement, any equilibrium outcome is achievable with a Nash equilibrium where an agreement has been reached. This means we only need to determine when there is an incentive for an individual decisionmaker to refuse to make an agreement, by changing the message sent in the negotiation process and choosing some other strategy from the elemental game to play. Given any beta-individually rational outcome, for each individual the opposing coalition can give strategy choices (using the f functions) which block the individual from raising his payoff, and these deterring threats can be used effectively in this game with communication since any deviating individual must, in effect, announce his intention to deviate in the negotiation process by announcing he will not agree to this outcome. Given this beta-individually rational outcome, an equilibrium results in the game with communication and binding agreements if the decisionmakers all send the same message of a selection yielding this outcome, play the blocking choices for another player whenever he refuses to agree and everyone else does agree, and play the strategy from the proposed agreement otherwise. When any potentially deviating player considers changing his message from a beta-individually rational outcome, he notes that any response to the blocking choices made by the others can not increase his payoff, so that there is no incentive to change his strategy unilaterally in this game with communication. This result supports the story told to justify the beta-individually rational outcomes as a cooperative solution concept, and it provides a rationale for using the beta definition over the alpha definition for the individually rational concept.

The analagous result when considering the power of coalitions, instead of the power of individuals, is not true for all games. That is the set of strong equilibrium outcomes in this game with communication is not necessarily identical with the beta-core of the associated elemental game. There is, however, a large class of games for which this result is true. Basically, for this result to hold there must be enough information revealed during the negotiation process to reveal which coalition will attempt to improve upon the outcome resulting from the agreement, so that the opposing coalition can form and block it, or the game has payoffs where the same blocking strategies are effective against several coalitions.

As with the result for individuals, we see directly from the definitions that:

Theorem 3: Any strong equilibrium outcome from any game with communication must be in the beta-core of the associated elemental game.

Of course, as with individuals, it is more interesting to know when the set of strong equilibrium outcomes from the game with communication and binding agreements is big enough to equal the beta-core of the associated cooperative elemental game.

When considering the power of coalitions, one important coalition is the coalition of all players. Here, we say an outcome is Pareto optimal if the coalition of all players cannot improve upon the outcome. This coalition is important here because the opposing coalition is empty, so that no information is needed from the negotiation process to capture the same strategic opportunities in the game with communication as are described by our story. Here, we immediately have from the definitions:

Lemma 4: An outcome is Pareto optimal in the elemental game if and only if it is Pareto optimal in any associated game with communication.

Combining this result with Theorem 2 we have

Theorem 5: For any two-person elemental game, the set of strong equilibrium outcomes for the game with communication and binding agreements is identical to the beta-core of the cooperative elemental game.

In games with more than two players, there may be some beta-core outcomes in the elemental game which are not strong equilibrium outcomes in the game with communication and binding agreements. An example is the three-person game where each player can choose either a zero or a one and the payoffs are given by the following table:

#3 chooses 0

#1 \ #2	0	1
0	2,0,0	0,0,0
1	1,0,2	0,0,1

#3 chooses 1

#1 \ #2	0	1
0	1,1,2	0,0,0
1	0,0,0	0,0,0

In this game the beta-core is {(2,0,0), (1,0,2), (1,1,2)} and the set of strong equilibrium outcomes from the associated game with communication and binding agreements is {(1,0,2), (1,1,2)}. For us, the important outcome to examine is (2,0,0).[7] The outcome (2,0,0) is in the beta-core, as no coalition can beta-improve upon it. #1 is already attaining his highest payoff; #2 cannot change his payoff with a unilateral change in his strategy; {#3} is blocked if #1 chooses a zero and #2 chooses a one; and {#2, #3} is blocked if #1 chooses a one. If the outcome (2,0,0) were a strong equilibrium outcome in the game with communication and binding agreements, then there would be a strong Nash equilibrium where all agreed to this outcome, each sending a message of (0,0,0). If player #1 specifies with f^1 that he will play a one in any case where #3 changes his message and the others do not change their messages, #3 gains by just changing his message. If player #1 specifies he will play a zero in any of these cases, then {#2, #3} gain by claiming the outcome (1,1,2). Thus, given any selection of strategies in this game with communication which yields (2,0,0), player #3 has an incentive to change his strategy (possibly with player #2), so that this outcome cannot be a strong equilibrium outcome in the game with communication and binding agreements.

In effect, when player #3 changes his message and the others do not change their messages, player #1 does not know whether to respond with his strategy to block {#3}, choosing a zero, or with his strategy to block {#2, #3}, choosing a one. With these messages the players know only that some superset of {#3} is attempting to improve its payoff. In this case, players #1 and #2 can block {#3}, but we argue #2 will not choose to do so as it is not in his self-interest. By refusing to block {#3} he can increase his own payoff.

We have a situation in this example where for some $\pi(\bar{s})$ in the beta-core a selection s_C exists where C is a particular coalition, so that for all $s^C \in S^C$, $\pi^i(s_C/s^C) \leq \pi^i(\bar{s})$ for some $i \in C$, that is s_C blocks the coalition C from improving upon $\pi(\bar{s})$. However, for any blocking s_C there is some $K \supset C$ and an $s^K \in S^K$ where $\pi^i(s_C/s^K) > \pi^i(\bar{s})$ for all $i \in K$. If some members of the opposing coalition join the coalition C, then the new coalition K can improve upon its payoff given that those remaining in the opposing coalition are still trying to block C. In this situation it is not rational for those players in K but not in C to participate in an attempt to block C, since refusing to block improves their payoff.

This notion may be used to modify the concept of the core. Up until now we have been comparing the strong equilibrium outcomes of this game with communication to the beta-core, and by definition $\pi(\bar{s})$ is in the beta-core if for all coalitions C there is an $s_C \in S$, such that for all $s^C \in S^C$, $\pi^i(s_C/s^C) \leq \pi^i(\bar{s})$ for some $i \in C$. We will modify the definition of the core and say $\pi(\bar{s})$ is in the hard-core if for all coalitions C there is an $s_C \in S$, such that for all $s^K \varepsilon S^K$ where $K \supset C$, $\neq C$, $\pi^i(s_C/s_K^K) \leq \pi^i(\bar{s})$ for some $i \varepsilon K$. Instead of requiring that all coalitions can be blocked from improving upon their payoffs, we are requiring that all coalitions can be rationally blocked from improving upon their payoffs. With this modification we have:

Theorem 6: The set of strong equilibrium outcomes in the game with communication and binding agreements is identical to the hard-core of the cooperative elemental game.

Luckily for our applications, the beta-core and the hard-core are often identical. For example, say that for any $\pi(\bar{s})$ in the beta-core there is a selection $d \varepsilon S$ such that for all C and all $s^C \varepsilon S^C$, $\pi^i(d/s^C) \leq^* \pi^i(\bar{s})$ for some $i \varepsilon C$. Since the same selection is used as a deterrent for all coalitions, $\pi(\bar{s})$ is also in the hard-core. In economic applications, this deterrent may be using the competitive strategies in an oligopoly market, purchasing nothing in a public goods market, or refusing to trade in an exchange economy.

CONCLUSION

The major goal of this paper was to construct a game where the essential elements of the stories told about pre-game communication are made explicit and to compare the equilibrium outcomes of this game with communication to the different solutions proposed for the elemental game where these elements are only implicit. We found that the equilibrium outcomes of a noncooperative game are unaffected by the negotiation process. No non-binding communication has an effect on the equilibrium outcomes. This both supports the story told about the effect of a pre-game negotiation process on noncooperative equilibria and justifies our use of messages which only include the potentially binding communication. We found that the equilibrium outcomes of the game with communication and binding agreements are identical to the beta-individually rational outcomes. We also found that the strong equilibrium outcomes of this game with communication are identical to the hard-core, a modified version of the beta-core. These results largely support the stories told about the effect of pre-game communication on cooperative games and provide a rationale for using the beta definitions over the alpha definitions for individual rationality and the core.[8] We also introduced the hard-core, a modified version of the beta-core, that is necessary to use when the identity of the coalitions which are attempting to improve their payoffs are not fully revealed by the negotiation process. There is some discussion on when the beta-core may differ from the hard-core.

The structure of the game with communication seems to lend itself to many future refinements. If the core concept gives too many outcomes to be useful in a particular application, the equilibrium concept can be refined using a perfect equilibrium concept[9] or using intermediate sized penalties where the enforcement of agreements may not be perfect. If the core concept gives too few outcomes to be useful in a particular application, more outcomes can be added by considering costs of decisionmaking using ε - equilibria[10] or by adding costs of communicating within a coalition. A major advantage of using the structure provided by a game with communication is that it eliminates the ad hoc nature of the solution concepts used in these extensions.

APPENDIX

Proof of Theorem 1: (i) Show that if $\sigma = (m,f)$ is a (strong) Nash equilibrium in the game with communication and no binding agreements, then $f(m)$ is a (strong) Nash equilibrium in the elemental game. Using the contrapositive, if there is a coalition C I with strategies s^C such that $\pi^i(f(m)/s^C) > \pi^i(f(m))$ for all $i \epsilon C$, then for any $\bar{\sigma}^C = (m^C, \bar{f}^C)$ where $\bar{f}^C(m) = s^C$, $\Pi^i(\sigma/\bar{\sigma}^C) > \pi^i(f(m)) = \Pi^i(\sigma)$ for all $i \epsilon C$. Thus, any (strong) equilibrium outcome in the game with communication and no binding agreements is a (strong) equilibrium outcome in the elemental game.

(ii) Show that if is s a (strong) Nash equilibrium in the elemental game, then $\sigma = (m,f)$, where $f(\bar{m}) = s$ for all $\bar{m}$, is a (strong) Nash equilibrium in the game with communication and no binding agreements. Using the contrapositive, if $\sigma = (m,f)$ where $f(\bar{m}) = s$ for all $\bar{m}$ and there is a coalition $C \subset I$ with stragegies $\bar{\sigma}^C = (\bar{m}^C, \bar{f}^C)$ such that $\Pi^i(\sigma/\bar{\sigma}^C) > \Pi^i(\sigma)$ for all $i \epsilon C$, then $\pi^i(\bar{f}^C(m/\bar{m}^C)) = \Pi^i(\sigma/\bar{\sigma}^C) > \Pi^i(\sigma) = \pi^i(s)$ for all $i \epsilon C$. Thus, any (strong) equilibrium outcome in the elemental game is a (strong) equilibrium outcome in the game with communication and no binding agreements.

Proof of Theorem 2: (i) Show that if $\sigma = (m,f)$ is a Nash equilibrium in the game with communication and binding agreements, then $\pi(f(m))$ is beta-individually rational for the elemental game. Using the contrapositive, if there is an individual $i \varepsilon I$ where for all $s \varepsilon S$ there is a strategy $\bar{s}^i \varepsilon S^i$ such that $\pi^i(s/\bar{s}^i) > \pi^i(f(m))$, then there is a $\bar{\sigma}^i = (\bar{m}^i, \bar{f}^i)$ where $\bar{m}^i \neq m^i$ and $\bar{f}^i(m/\bar{m}^i) = \bar{s}^i$ such that

$$\Pi^i(\bar{\sigma}/\sigma^i) = \pi^i(f(m/\bar{m}^i)/\bar{s}^i) > \pi^i(f(m)) \geq \Pi^i(\sigma).$$

Thus, any equilibrium outcome for the game with communication and no binding agreements is a beta-individually rational outcome for the elemental game.

(ii) Show that if $\pi(s)$ is beta-individually rational for the elemental game, then there is a $\sigma = (m,f)$ where $f(m) = s$ and which is a Nash equilibrium in the game with communication and binding agreements. Given such an s, this means for all individuals $j \varepsilon I$ there is an $s_j \varepsilon S$ where for all $\bar{s}_j \varepsilon S^j$, $\pi^j(s_j/\bar{s}^j) \leq \pi^j(s)$. Consider the selection $\sigma = (m,f)$ where for all $i \varepsilon I$, $m^i = s$ and

$$f^i(\bar{m}) = \begin{cases} s^i_j & \text{if } \bar{m}^j \neq s \text{ and } \bar{m}^k = s \text{ for all } k \neq j \text{ (all } i \varepsilon I) \\ s^i & \text{otherwise.} \end{cases}$$

We will show that σ is a Nash equilibrium in the game with communication and binding agreements.

Showing this by contradiction, assume σ is not a Nash equilibrium in the game with communication and binding agreements. This means there is an individual $i \varepsilon I$ with a stragegy $\bar{\sigma}^i = (\bar{m}^i, \bar{f}^i)$ such that

$\Pi^i(\sigma/\bar{\sigma}^i) > \Pi^i(\sigma) = \pi^i(s)$. This implies $\bar{m}^i \neq s$

and $\Pi^i(\sigma/\bar{\sigma}^i) = \pi^i(f(m/\bar{m}^i) \;/\; \bar{f}^i(m/\bar{m}^i))$

$$= \pi^i(s_i \;/\; \bar{f}^i(m/\bar{m}^i))$$

$$> \pi^i(s),$$

contradicting the statement that for any

$f^i(m/\bar{m}^i) \varepsilon S^i$, $\pi^i(s_i \;/\; f^i(m/\bar{m}^i)) \leq \pi^i(s)$.

Thus, σ must be a Nash equilibrium in the game with communication and binding agreements, demonstrating that any beta-individually rational outcome for the elemental game is an equilibrium outcome for the game with communication and binding agreements.

<u>Proof of Theorem 3</u>: Show that if $\sigma = (m,f)$ is a strong Nash equilibrium in a game with communication, then $\pi(f(m))$ is in the beta-core of the elemental game. Using the contrapositive, if there is a coalition $C \subset I$ where for all $s \varepsilon S$ there are strategies $\bar{s}^C \;\varepsilon\; S^C$ such that

$\pi^i(s/\overline{s}^C) > \pi^i(f(m))$ for all $i\varepsilon C$, then for coalition C there exist strategies $\overline{\sigma}^C = (\overline{m}^C, \overline{f}^C)$ where $\overline{m}^C \neq m^C$ and $\overline{f}^C(m/\overline{m}^C) = \overline{s}^C$ such that

$\pi^i(\sigma/\overline{\sigma}^C) = \pi^i(\overline{f}(m/\overline{m}^C) / \overline{s}^C) > \pi^i(f(m)) \geq \Pi^i(\sigma)$

for all $i\varepsilon C$.

Thus, any strong equilibrium outcome in a game with communication is in the beta-core of the elemental game.

Proof of Lemma 4: (i) Show that if $\pi(s)$ is Pareto optimal in the elemental game, then $\sigma = (m,f)$, where m^i=s for all i and f(m)=s, is Pareto optimal in any game with communication. If such a σ is not Pareto optimal, there is a $\overline{\sigma}=(\overline{m},\overline{f})$ such that m≠ and $\Pi(\overline{\sigma}) = \pi^i(\overline{f}(\overline{m})) > \pi^i(s)=\Pi^i(\sigma)$ for all i, so that $\pi(s)$ is not Pareto optimal in the elemental game.

(ii) Where $\sigma=(m,f)$, show that if $\Pi(\sigma)$ is Pareto optimal in a game with communication, then $\pi(f(m))$ is Pareto optimal in the elemental game. If $\pi(f(m))$ is not Pareto optimal, there is an s such that $\pi^i(s)>\pi^i(f(m))$ for all i and for $\overline{\sigma} = (\overline{m},\overline{f})$ where $\overline{m}\neq m$ and $\overline{f}(\overline{m})$=s,

$\Pi^i(\overline{\sigma}) = s$, $\pi^i(s) > \pi^i(f(m)) \geq \Pi^i(\sigma)$ for all i, so that $\overline{\sigma}$ is not Pareto optimal.

Proof of Theorem 5: This follows immediately from the proofs of Theorem 2 and Lemma 4.

Proof of Theorem 6: (i) If $\pi(\bar{s})$ is in the hard-core, then for all C there is an $s_C \varepsilon S$ such that for all $s^K \varepsilon S^K$ where $K \supset C$, $\pi^i(s_C/s^K) \leq \pi^i(\bar{s})$ for some $i \varepsilon K$. Consider $\sigma=(m,f)$ where $m^i=\bar{s}$ for all i and

$$f^i(m) = \begin{cases} s^i_C & \text{if } (\forall j \varepsilon C)\ m^j \neq s \text{ and } (\forall j \notin C)\ m^j = s \\ s^i & \text{otherwise.} \end{cases}$$

We will show that σ is a strong Nash equilibrium. If not there are coalitions C and K with $\bar{\sigma}^K = (\bar{m}^K, \bar{f}^K)$ such that $(\forall j \varepsilon C) m^j \neq s$ and $(\forall j \notin C)\ m^j = s$, and

$\Pi^i\ (\sigma/\bar{\sigma}^K) = \pi^i(s_C/\bar{f}^K(\bar{\bar{m}}/\bar{m}^K)) > \pi^i(s) = \Pi^i(\sigma)$

for all $i \varepsilon K$, yielding a contradiction with the assumption that $\pi(\bar{s})$ is in the hard-core.

(ii) If $\pi(\bar{s})$ is not in the hard-core, then for some C it must be that for all $\bar{\bar{s}} \varepsilon S$ there is an $s^K \varepsilon S^K$ where $K \supset C$ and $\pi^i(\bar{\bar{s}}/s^K) > \pi^i(\bar{s})$ for all $i \varepsilon K$. Say $\sigma=(m,f)$ where $f(m)=\bar{s}$. Consider $\bar{\sigma}^K=(\bar{m}^K, \bar{f}^K)$ where $(\forall j \varepsilon C)\ m^j \neq \bar{s}$ and $(\forall j \notin C)\ m^j = \bar{s}$, and (choosing the appropriate s^K for $\bar{\bar{s}}=f(m/\bar{m}^k)$) $\bar{f}^K(m/\bar{m}^K) = s^K$. Then $\Pi^i(\sigma/\bar{\sigma}^K) = \pi^i(\bar{\bar{s}}/s^K) > \pi^i(\bar{s}) \geq \Pi^i(\sigma)$ for all $i \varepsilon K$, so that σ is not a strong Nash equilibrium.

FOOTNOTES

*Helpful discussions on this topic are acknowledged with John Roberts and Jim Cox. Of course, I retain ultimate responsibility for any errors.

1. $S^C \equiv \underset{i \varepsilon C}{X} S^i$ is the set of feasible strategies for each coalition $C \subset I$, and $S \equiv S^I$

2. To simplify the notation let $\sigma \equiv (\sigma^i)_{i \varepsilon I}$, $m \equiv (m^i)_{i \varepsilon I}$, and $f \equiv (f^i)_{i \varepsilon I}$. Also, for any coalition $C \subset I$ let $\sigma^C \equiv (\sigma^i)_{i \varepsilon C}$, $m^C \equiv (m^i)_{i \varepsilon C}$, and $f^C \equiv (f^i)_{i \varepsilon C}$.

3. The slash (/) indicates a substitution of variables, e.g. $\Pi^i(\sigma/\bar{\sigma}^i) = \Pi^i(\sigma^1,\ldots,\sigma^{i-1},\bar{\sigma}^i,\sigma^{i+1},\ldots,\sigma^n)$ if $I = \{1,\ldots,n\}$.

4. All proofs are in the Appendix. All of the proofs are a relatively straightforward application of the definitions, and are included primarily for completeness.

5. For a discussion of these two definitions see Shapley and Shubik (1973) or Aumann (1967).

6. "Whether the alpha-notion or the beta-notion is preferable is a matter of taste... The alpha-notion seems to be intuitively more appealing, but as we shall see the beta-notion has a certain technical advantage." Aumann (1967), p. 20. See footnote 7 for a description of this technical advnatage.

7. As for the other outcomes, (0,0,0) and (0,0,1) do not belong to any of these solutions as they are not Pareto optimal. (1,0,2) is a strong equilibrium outcome in this game with communication as the players can agree on this outcome, #3 can threaten to play a one if #1 or #2 change their messages, and they can play the agreed upon strategies otherwise; (1,0,2) is not a strong equilibrium outcome for the elemental game as player #1 has an incentive to change his strategy unilaterally to a zero. (1,1,2) is a strong equilibrium outcome for any game with communication, and thus belongs to the beta-core, as no coalition has an incentive to change its strategies unilaterally.

FOOTNOTES (cont.)

8. These results are closely related to Theorem 13 in Aumann (1967), that the beta-core of a game coincides with the strong equilibrium outcomes of its supergame. As constructed by Aumann, the supergame can be analyzed to determine the effect of threats which can be used in a dynamic environment with no discounting. We can then compare the equilibria achieved with experience in a dynamic environment to the equilibria achieved with communication in a static setting.

9. See Selten (1975) and Aumann (1967).

10. See Alger (1979).

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