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THE LENGTH OF OPTIMAL EXTRACTION PROGRAMS WHEN DEPLETION AFFECTS EXTRACTION COSTS[+]

by

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A distinguishing feature of nonrenewable resource production is that the marginal cost of extracting the resource at a given rate increases with previous cumulative extraction.[1] For example, in the case of oil, extraction costs increase from having to compensate for a loss of natural pressure in the well. In the case of minerals, lower grade ores or less accessible deposits must be utilized as the resource is depleted. Mineral resources have an additional characteristic of interest. Scarcer elements, like copper, zinc, and silver occur in several distinct chemical structures. As a result, when the next most difficult structure must be confronted, the unit cost of extraction rises discontinuously.[2]

This paper considers how these realistic aspects of resource recovery affect the length of an optimal extraction program. For concreteness we assume there is some amount, $\bar{I} < \infty$, of the resource that is ultimately available for extraction. Let S_i be the stock remaining at the beginning of period i and let q_i be the amount of extraction during period i. The cost of extraction is given by $C(q_i, S_i)$. Denote $\tilde{C}(S)$ as the average cost of extracting at an infinitesimal rate with S units remaining: $\tilde{C}(S) = \lim_{q \to 0} C(q,S)/q$. Then, according to the discussion above, this limiting unit extraction cost increases as S decreases. Also, it may rise discontinuously when distinct deposits [illegible] lower cost resources are exhausted. This phenomenon is illustrated [illegible] Figure 1a with the jumps in unit costs occurring at stocks S^1 and S^2.

Figure 1b illustrates our assumptions about the demand for the resource. The inverse demand function, $P(q)$, is assumed to be

Figures la and lb go here.

stationary, downward sloping, and to intersect the price axis at a finite "choke" price denoted by P(0).[3]

In an optimal extraction program, unit extraction costs rise over time and price approaches the choke level. The relationship between the choke price P(0) and the unit extraction cost in the neighborhood of P(0) turns out to be important in determining the length of the optimal extraction program. We show that the resource is optimally extracted in infinite (finite) time, if unit extraction cost rises continuously (discontinuously) for values of $\tilde{C}$ just below P(0). For example a choke price equal to P_5 or P_2 or P_1 in Figure la will result in an optimal program of infinite length. A choke price equal to P_4 or P_3 will cause the time before optimal extraction ceases to be finite.

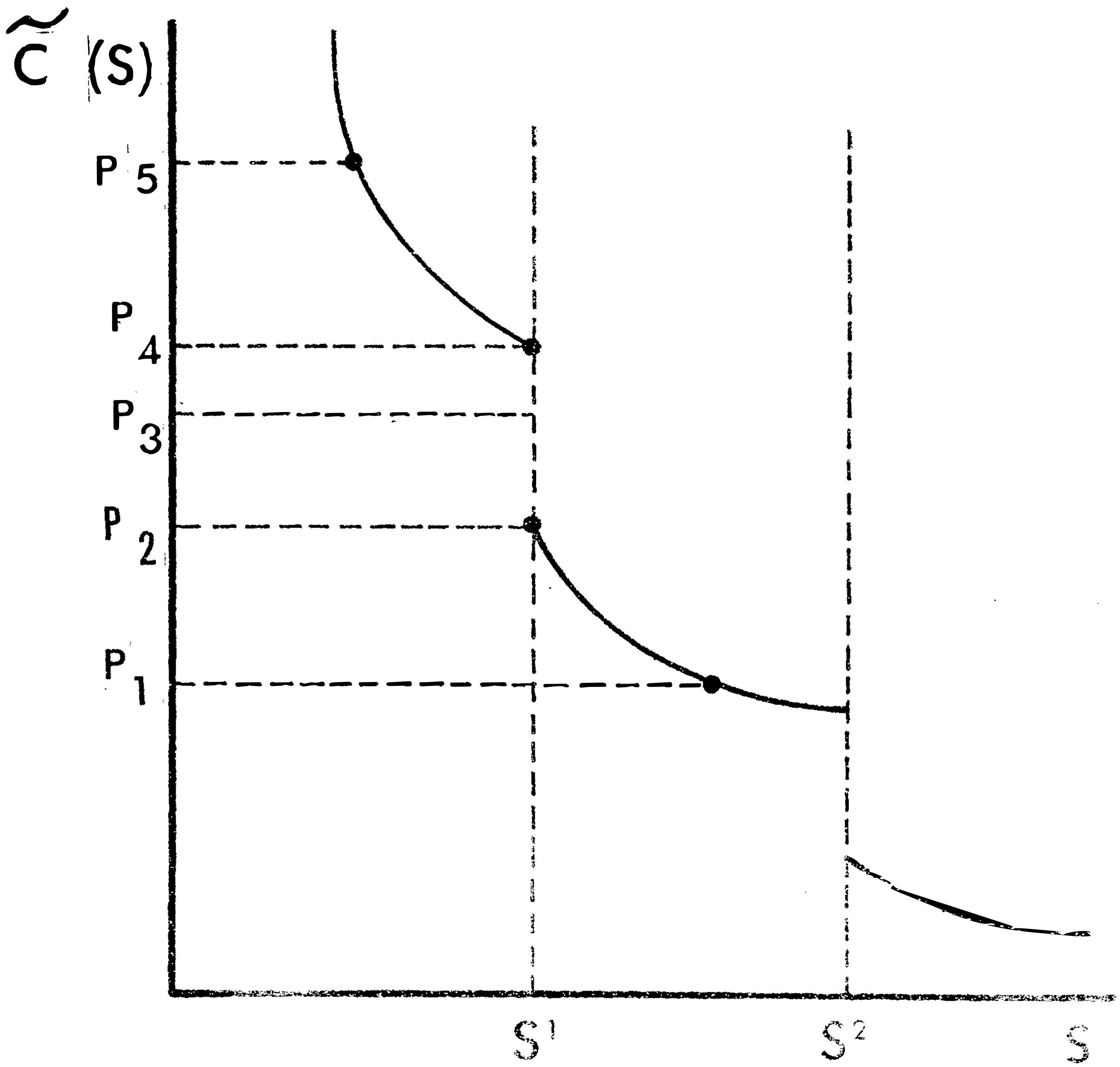

Figure 1a

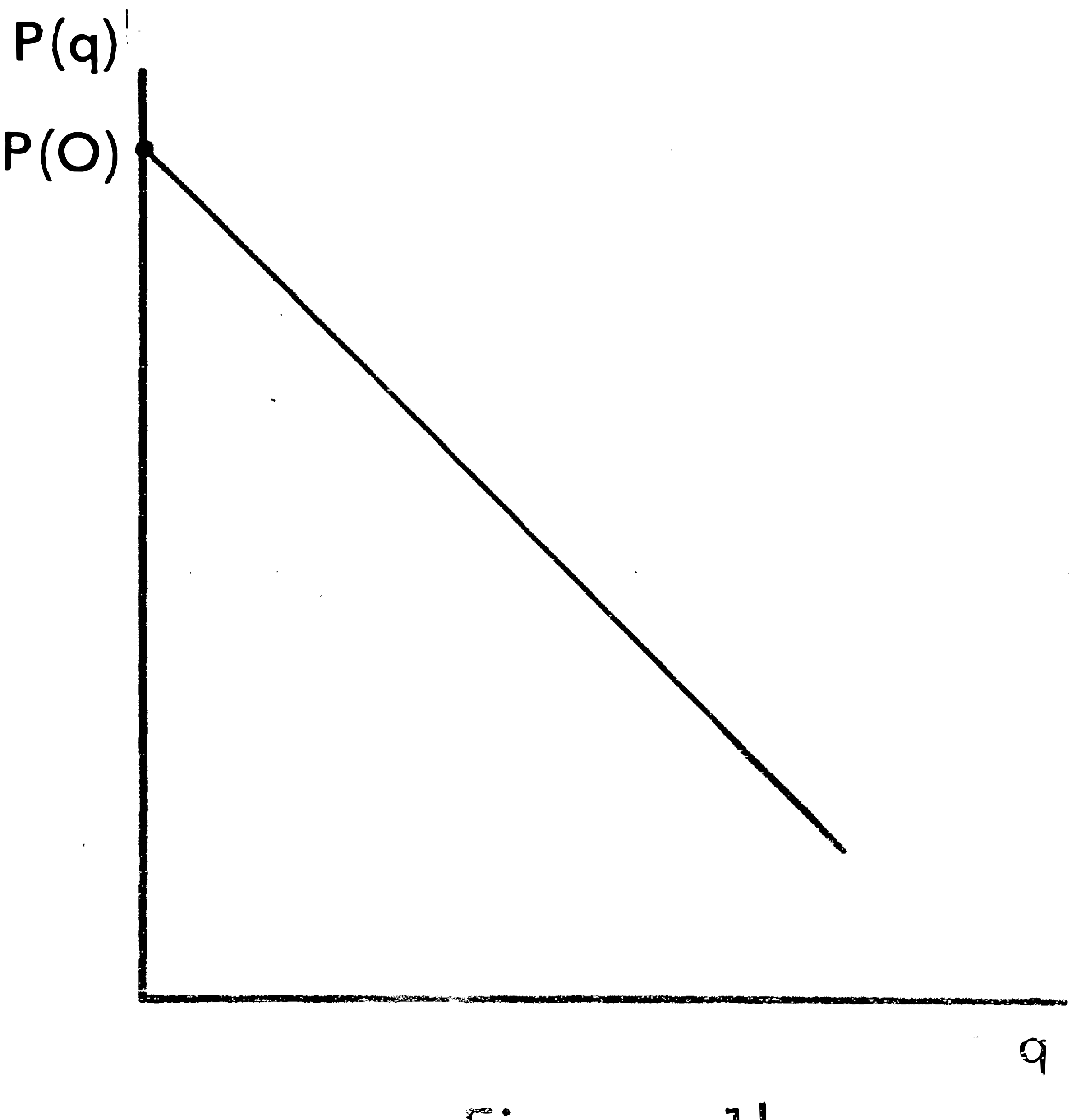

Figure 1b

This result is noteworthy on four counts. First, it contradicts a general belief that optimal extraction of a finite resource requires infinite time <u>only</u> in the "neither believable nor important" case noted by Solow[4] where the choke price is infinite. Second, it is currently popular to talk about the availability of a natural resource in terms of the remaining number of years that it can be economically exploited. Our result points out an undesirable feature of this measure of resource abundance. It may be quite sensitive to minor fluctuations in demand. A shift in demand which causes a small change in the choke price, (a drop from P_3 to P_2, say) can produce dramatic changes in the economic lifetime of the resource. Furthermore, the cumulative amount of the resource that is ultimately recovered and the length of the extraction horizon need not even be correlated. For example, ultimate recovery increases but the extraction time becomes shorter when the choke price shifts from P_1 to P_3.

Third, our result has important implications for the way one represents depletion effects in models of resource extraction. Some studies of resource industries approximate a continuously rising unit cost of extraction curve, $\tilde{C}(S)$, by a step function. This kind of approximation is represented in Figure 2 by the dotted line.[5] The step function approximation is handy because it reduces the difficulty of computing the optimal extraction program. However, at least one characteristic of the optimal program--its duration--is badly approximated by the step function approach. With the step function, extraction costs [illegible] increase in jumps. Hence, our results imply that no matter how fine the approximation, it always involves terminating extraction in finite time, even if the situation being approximated requires an infinite extraction horizon.

Figure 2 goes here.

Finally, our results put in perspective the traditional analysis where depletion effects are neglected. In the absence of depletion effects, the extraction problem can be posed in either of two equivalent ways: If the mine initially contains $\bar{I}$ units, this fact can be summarized by a constraint on cumulative extraction or, alternatively, by a continuous cost function whose limiting unit cost jumps above the choke price once $\bar{I}$ units have been extracted. The latter approach permits us to apply our analysis and to conclude that optimal extraction must terminate in finite time.

To demonstrate our results we assume the extractor is a monopolist making decisions at discrete intervals. The same results obtain, however, if the extractor were instead a planner, or if decisions were made continuously. Continuous time results are derived in the Appendix.

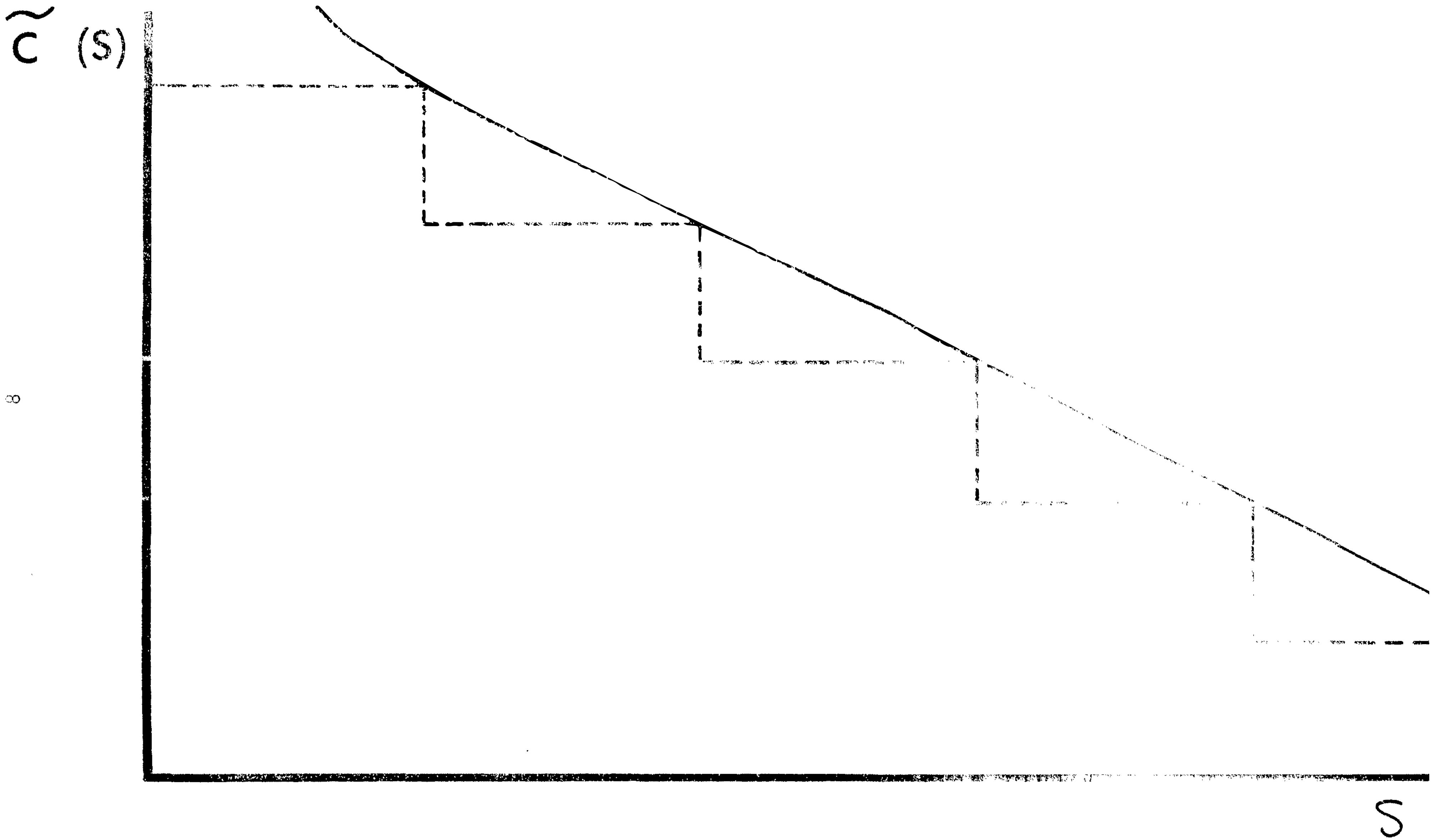

Figure 2

Formally the monopolist's problem is to choose $\{q_1, q_2, \ldots\}$ in order to

$$\text{maximize} \quad \sum_{i=0}^{\infty} B^i [R(q_i) - C(q_i, S_i)] \qquad (1)$$

subject to

$$S_i \geq q_i \geq 0$$

$$S_o = \bar{I} < \infty$$

$$S_{i+1} = S_i - q_i$$

where $R(q_i)$ is the revenue produced by q_i and B is a constant discount factor.

It can be shown[7] that a maximum always exists for such a problem provided that,

(a) the net profit function, R − C, is continuous[8] and that it is bounded from above,

(b) $0 \leq B < 1$.

It is assumed that these weak restrictions are satisfied by the problem under consideration. Additional assumptions must be made about the net profit functi[illegible] [illegible] generate our results. We require that

A1. $R(0) - C($[illegible]$) = 0$;

A2. $\exists\ S^* \in ()$, [illegible] that $S^* = \inf (S \mid R(q) - C(q,$ [illegible]$) < 0,\ q >$ [illegible]$)$;

A3. $R_q(q)$ − [illegible] (wherever it is defined) is strictly increasing in [illegible]

A4. $R_q(q)$ − [illegible] (q,S), (wherever it is defined) is strictly decreas[illegible]

These assumptions are reasonable.[9] Since the revenue from selling (and the cost of extracting) nothing is zero, the first assumption is plausible. The second simply defines some level beyond which extraction is too costly. The third asserts the existence of a depletion effect which shifts the marginal cost curve up as the stock remaining falls. The last is slightly stronger than the standard assumption that marginal profits are a downward sloping function of the rate of extraction although as indicated below, the weaker formulation is sufficient to prove our basic results in continuous time.

We require that the net profit function be continuous everywhere and twice continuously differentiable except at a finite set of points denoted Z. Z is the set of stocks S^j (S^1 and S^2 in Fig. 1a), where the limiting unit extraction costs rise discontinuously as S declines. We assume that for any q_t, S_t such that $S_t \notin Z$ and $S_t - q_t \notin Z$ the net profit function is twice continuously differentiable. This assumption permits us to consider most cases of interest.[10]

To prove our results we need to look at two possibilities,

$$P(0) - \lim_{S \downarrow S^*} \tilde{C}(S) = R_q(0) - \lim_{S \downarrow S^*} C_q(0,S) = 0 \qquad (2)$$

$$P(0) - \lim_{S \downarrow S^*} \tilde{C}(S) = R_q(0) - \lim_{S \downarrow S^*} C_q(0,S) > 0. \qquad (3)$$

The first possibility occurs at choke prices P_5, P_2 or P_1 in Figure 1a while the second possibility occurs at choke prices P_4 or P_3.

If (2) holds, we will show that the optimal terminal extraction time, T is infinite. Assume to the contrary that optimal extraction ceases in finite time, so that $T < \infty$. Clearly, $S_{T+1} > S^*$ is not optimal since (2) and (A3) would then imply that further extraction after time T would be profitable. Nor is $S_{T+1} < S^*$ optimal by (A2). Hence $S_{T+1} = S^*$. By hypothesis, T is the final period of extraction so that $q_T = S_T - S^* > 0$. The profit earned in that period is $R(q_T) - C(q_T, q_T + S^*)$. But (2) in conjunction with (A1) and (A4) implies this profit is strictly negative for any $q_T > 0$. Hence we can dominate any proposed program of this type by simply replacing the extraction in period T with abstention. Since a maximum exists and any finite program can be dominated if (2) holds, the optimal extraction horizon must be infinite.

If (3) holds, we will show that the optimal terminal extraction time, T is finite. For suppose it is claimed that an infinite program is optimal. The claim is obviously false if $S_t < S^*$ for some t or--given the stationarity of (1)--if $q_t = 0$ for some t. If, however, $q_t > 0$ and $S_t \geq S^*$ for all t then $\{S_t\}$ is a strictly decreasing sequence that lies in the interval $[S^*, \bar{I}]$. Thus $S_t \to \bar{S}$ as $t \to \infty$ for some $\bar{S} \in [S^*, \bar{I}]$ by the monotone convergence theorem. Moreover, since $\{S_t\}$ is a Cauchy sequence, $S_t - S_{t+1} = q_t \to 0$ as $t \to \infty$.

Consider the value of increasing q_t and decreasing q_{t+1} by an equal amount leaving unchanged the rest of the program:

$$\left.\frac{d\Pi}{dq_t}\right|_{S_{t+2}} = R_q(q_t) - C_q(q_t, S_t) - B[R_q(q_{t+1}) - C_q(q_{t+1}, S_{t+1}) - C_S(q_{t+1}, S_{t+1})] \quad (4)$$

wherever $S_i - q_i \notin Z$ and $S_i \notin Z$ for $i = t, t+1$.

If the infinite program proposed is optimal $\left.\frac{d\Pi}{dq_t}\right|_{S_{t+2}} = 0$ for all t for which the derivatives are defined.

Since Z is a finite set, there exists an interval $I = (\overline{S}, \hat{S})$ for $\overline{S} < \hat{S} \leq \overline{I}$ such that $S \in I$ implies $S \notin Z$ and $S - q \notin Z$. Thus, beyond some date the derivatives are well-defined. Since the choke price is finite, $\lim_{q \to 0} R_q(q)$ exists and we obtain:

$$\lim_{t \to \infty} \left. \frac{d\Pi}{dq_t} \right|_{S_{t+2}} = [R_q(0) - \lim_{S \downarrow \overline{S}} C_q(0,S)][1-B]. \tag{5}$$

Now by (3) and $B < 1$, (5) is strictly positive. Hence one can always find a date after which a dominant program can be constructed by expanding production in one period and cutting back by an equivalent amount in the following period. Since an optimal program exists and yet any infinite program can be dominated whenever (3) holds, it must be finite in this case.

APPENDIX

This Appendix establishes the continuous time analogues of the results presented in the text.

The problem facing the monopolist is to

$$\underset{\{q\},\ T}{\text{maximize}} \int_0^T e^{-rt}[R(q) - C(q,S)]dt \tag{1}$$

subject to $q,\ S,\ T \geq 0$, $S_o = \bar{I} < \infty$, and $\dot{S} = -q$. The finite set Z is defined as in the text. We assume that the profit function is twice continuously differentiable for $S \notin Z$, and that

B1. $R(0) - C(0,S) = 0$;

B2. $\exists\ S^* \in [0,\bar{I})$ such that $S^* = \inf(S \mid R(q) - C(q,S) < 0,\ q > 0)$;

B3. $R_q(q) - C_q(q,S)$, (wherever it is defined), is strictly increasing in S; and

B4. $R_q(q) - C_q(q,S)$, (wherever it is defined) is strictly decreasing in q.

The solution to (1), (assuming it exists), is characterized by the following necessary conditions:[11]

$$R_q(q) - C_q(q,S) - \lambda \leq 0\ (=, \text{ if } q > 0);\ S \notin Z. \tag{2a}$$

$$\dot{\lambda} = r\lambda + C_S(q,S);\ S \notin Z. \tag{2b}$$

$$\dot{S} = -q. \tag{2c}$$

$$H(T) = 0 \text{ for } T<\infty,\ \lim_{t\to T} H(t) = 0 \text{ for } T=\infty. \tag{2d}$$

$$e^{-rT}\lambda(T) \geq 0 \text{ for } T<\infty,\ \lim_{t\to T} e^{-rt}\lambda(t) = 0 \text{ for } T=\infty.\text{[12]} \tag{2e}$$

$$e^{-rT}\lambda(T)[\bar{I}-S(T)] = 0 \text{ for } T<\infty.$$

$$\lim_{t\to T} e^{-rt}\lambda(t)[\bar{I}-S(t)] = 0 \text{ for } T=\infty, \tag{2f}$$

where $H(t) = e^{-rt}[R(q) - C(q,S) - \lambda q]$.

Consider our first case where

$$R_q(0) - \lim_{S\downarrow S^*} C_q(0,S) = 0. \tag{3}$$

Assume $T<\infty$. It follows from (B2) that $S(T) \geq S^*$. Further (3) and (B3) imply $R_q(0) - C_q(0,S) > 0$ for $S > S^*$. This together with (B1) implies

$$S(T) = S^* < \bar{I}. \tag{4}$$

If $T<\infty$, then (4) and (2f) imply

$$\lambda(T) = 0. \tag{5}$$

Equations (3) - (5), (2a) and condition (B4) imply

$$q(T) = 0. \tag{6}$$

Since Z is finite there exists an interval $(S^*,\hat{S}]$ such that $S \in (S^*,\hat{S}]$ implies $S \notin Z$. Thus the optimal program is completely described by (2a) - (2f) as $S\downarrow S^*$. Notice also that (1) is a stationary (with respect to time) problem. Hence we can use (2a) - (2f) to represent the solution in synthesized form as a function of S. In particular, as $S\downarrow S^*$

$$\dot{S} = -q(S,\lambda(S)) = \dot{S}(S) \tag{7}$$

where q is implicitly defined in (2a) with

$$\delta q/\delta S = C_{qS}/(R_{qq} - C_{qq}) > 0$$

$$\delta q/\delta\lambda = 1/(R_{qq} - C_{qq}) < 0 \tag{2a'}$$

$$d\lambda/dS = \dot{\lambda}/\dot{S}.$$

We now establish that the nonlinear autonomous system $\dot{S}(S)$ takes infinite time to move from any initial position in $(S^*,\hat{S}]$ to S^*. The argument proceeds as follows. We first verify that $\dot{S}(S)$ has a finite first derivative at S^* and is continuous in some neighborhood around S^*. Accordingly $K(S^*-S) \leq \dot{S}(S) \leq 0$ for some $K \in (0,\infty)$. We next note that any linear system $\dot{S} = K(S^*-S)$ takes infinite time to decline to S^*. Since our nonlinear system declines more slowly than some linear system with stationary point S^*, it must take infinite time to reach S^*.

To begin, we verify that $\frac{d\dot{S}}{dS}(S^*)$ is bounded. Define $\hat{q}(S) = -\dot{S}(S)$.

Then, from (7), (2a'), and (2b) - (2c):

$$\frac{d\hat{q}}{dS} = \frac{\partial q}{\partial S} + \frac{\partial \hat{q}}{\partial \lambda}\frac{d\lambda}{dS} = \frac{+1}{R_{qq} - C_{qq}}\left[C_{qS} + \frac{r\lambda + C_S(q,S)}{-q}\right]. \tag{8}$$

Since $q \to 0$, $\lambda \to 0$ as $S \downarrow S^*$, the second term in square brackets in (8) is of indeterminate form, 0/0, as $S \downarrow S^*$. Applying L'Hospital's rule to evaluate the second term we obtain

$$\frac{\lim\limits_{S\downarrow S^*} \frac{d}{ds}(r\lambda + C_S)}{\lim\limits_{S\downarrow S^*} \frac{d}{ds}(-q)} = \frac{d\hat{\lambda}}{dS} = \frac{r\frac{d\hat{\lambda}}{dS} + \hat{C}_{SS} + \hat{C}_{Sq}\frac{d\hat{q}}{dS}}{-\frac{d\hat{q}}{dS}} \tag{9}$$

where the "hats" denote that variables are evaluated in the $\lim\limits_{S\downarrow S^*}$.

From (8) we solve for $d\hat{\lambda}/dS = d\hat{q}/dS(\hat{R}_{qq} - \hat{C}_{qq}) - \hat{C}_{qS}$.

Substituting for $d\hat{\lambda}/dS$ into (9) we obtain

$$\frac{d\hat{q}}{dS} = \frac{1}{\hat{R}_{qq}-\hat{C}_{qq}}\left[\frac{r\left(\frac{d\hat{q}}{dS}(\hat{R}_{qq}-\hat{C}_{qq})-\hat{C}_{qS}\right) + \hat{C}_{SS} + \hat{C}_{Sq}\frac{d\hat{q}}{dS}}{-\frac{d\hat{q}}{dS}} + \hat{C}_{qS}\right]. \tag{10}$$

It is apparent that (10) can not be satisfied for $d\hat{q}/dS = \infty$ or 0 which establishes that $\lim_{S \downarrow S^*} \dot{S}(S)$ is bounded.

Next, we verify that $\dot{S}(S)$ is continuous in some neighborhood of S^*. Since $S(t)$ and $\lambda(t)$ are continuous in t and H is strictly concave in q, q is continuous in t and hence in S. But $\dot{S} = -q$, so $\dot{S}(S)$ is continuous. Hence, we can conclude that:

$$0 \geq \dot{S}(S) \geq K(S^*-S), \text{ for some } K > 0 \text{ and any } S \in (S^*,\hat{S}]. \tag{11}$$

That is, our nonlinear autonomous system declines more slowly toward S^* than some autonomous linear system with the same stationary point. Hence (11) and the fact that $S(t) = S(0) + \int_0^t \dot{S}(u)du$ imply

$$S(t) \geq S^* + e^{-Kt}(S_o - S^*). \tag{12}$$

But (12) implies $S(t) > S^*$ for all $t<\infty$. This violates our original supposition that $T<\infty$ and that S reaches S^* in finite time. Thus $T=\infty$.

Consider now the second case where

$$R_q(0) - \lim_{S \downarrow S^*} C_q(0,S) > 0. \tag{13}$$

Assume $T=\infty$. The stationarity of (1) implies $q(t) > 0$ for all $t<T$ since it is never optimal to have an interval of zero production followed by an interval of positive production. Clearly $\lim_{t \to T} S(t) \geq S^*$ by (B2). Since the total amount ultimately extracted, $\bar{I} - \lim_{t \to T} S(t)$, is finite it follows that

$$\lim_{t \to T} q(t) = 0; \tag{14}$$

otherwise there would exist a t^* and an $\varepsilon > 0$ such that $q(t) \geq \varepsilon$ for $t>t^*$ and cumulative extraction would be finite.

Now since $q(t) > 0$, (2a) implies that $\lambda = R_q(q) - C_q(q,S)$. But since the choke price is finite and marginal profits strictly decrease in q and increase in S, $\lambda < R_q(0) - C_q(0,\bar{I}) < \infty$.

Since $C(0,S) = 0$, it follows that $C_S(0,S) = 0$ so that from (2b), $\dot{\lambda} \sim r\lambda$ as $t \to T$ and $q(t) \to 0$. Since λ is bounded and yet $\dot{\lambda} \sim r\lambda$ as $t \to T$,

$$\lim_{t \to T} \lambda(t) = 0. \tag{15}$$

Together (14),(15), and (2a) imply

$$R_q(0) - \lim_{S \downarrow S(T)} C_q(0,S) \leq 0. \tag{16}$$

But (13), (B3) and the fact that $S(T) \geq S^*$ imply a violation of (16). Hence our original presumption that $T = \infty$ cannot be correct, so that T must be finite.

FOOTNOTES

1/ For a discussion of this point, see Gordon (1967).

2/ See Brobst (1979).

3/ The last assumption is equivalent to saying that the resource is not essential.

4/ Typical of this view is the following remark made by Solow (1974).

> "The Age of Oil or Zinc or Whatever It Is will have come to an end. (There is a limiting case, of course, in which demand goes asymptotically to zero as price rises to infinity and the resource is exhausted only asymptotically. But it is neither believable nor important.)"

5/ This is formally equivalent to assuming that the resource base consists of a collection of pools with different, constant unit extraction costs. Solow and Wan (1976) use this convention to calculate optimal extraction programs.

6/ For the case of the planner, the net benefit function, $U(q,S)$, replaces the profit function of the monopolist in (1).

7/ To prove existence, the Weierstrass theorem may be applied. This requires generalizing the concepts of continuity of a function and compactness of its domain to infinite sequences. See Milton Harris [1978] for details.

8/ Note that we are assuming $C(q,S)$ is continuous, although C_q and C_S may not be.

9/ Notice that A1-A4 constitute restrictions on the net profit function rather than on either the revenue or cost function individually. Since A1-A4 can be translated into restrictions on the net benefit function ($U(q,S)$), it is clear that the basic result also applies to the planning problem.

10/ If the rate of extraction does not influence its cost, the extraction cost function can be written as $C(q,S) = g(S-q) - g(S)$, where $g(X)$ is a strictly decreasing, strictly convex function. It should be noted that this special case satisfies A3 and A4. Furthermore, if $g(S)$ is not differentiable at S', $C(q,S)$ is not differentiable at any (q,S) pair such that $S=S'$ or $S-q = S'$. This cost function has the following intuitive interpretation. Let $g(X)$ be the cost of drawing the mine down from its initial level to where X units remain. Then $g(S-q) - g(S)$ is the cost of drawing the mine down from its initial level to where S-q remains less the cost of drawing it down from its initial level to where S remains; hence, it is the cost of drawing the mine down from where S remains to where S-q remains.

11/ See Takayama (1974, theorem 8.C.3., p. 655).

12/ The classes of problems for which the infinite time transversality condition is necessary are discussed in Seierstad (1979). The proofs of our propositions do not utilize this condition.

REFERENCES

1. D. Brobst, "Fundamental Concepts for the Analysis of Resource Availability" in Scarcity and Growth Reconsidered, edited by V. Kerry Smith, Resources for the Future (1979), Johns Hopkins University Press, Baltimore.
2. R. Gordon, "A Reinterpretation of the Pure Theory of Exhaustion," Journal of Political Economy (1967).
3. M. Harris, "Dynamic Competitive Analysis" Carnegie-Mellon Working Paper No. (47-817), 1978.
4. A. Seierstad, "Transversality Conditions for Control Problems with Infinite Horizons," mimeo 1978.
5. R. Solow, "The Economics of Resources or the Resources of Economics," American Economic Review (May, 1974).
6. R. Solow and F. Wan, "Extraction Costs in the Theory of Exhaustible Resources," Bell Journal of Economics (Autumn, 1976).
7. A.Takayama, Mathematical Economics, Dryden Press (1974), Hinsdale, Illinois.