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LOSSES DUE TO MERGER

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Losses Due to Merger

In a previous note in this series, Salant, Switzer and Reynolds (1980) found that the intuitive notion that unity makes strength is not borne out in oligopoly models solved by the use of the Cournot-Nash noncooperative equilibrium concept. Specifically, they discovered that firms which merge may well be worse off after merger than they were when they operated as separate entities.

This raises a serious question for economists interested in game-theoretic analysis of oligopoly models, as well as a general question as to the separability of solution concepts and coalition structures. From the oligopoly theorist's point of view, this result could be seen as a criticism of the Nash Equilibrium solution for this class of games, or as evidence that "industry structure" and "industry behavior" are essentially interrelated.

The purpose of this note is to demonstrate that the latter interpretation is the best one for practical purposes. This is not to say that there are not severe conceptual and practical problems with Nash Equilibrium in this context. However, we shall demonstrate that most of the other "structure-free" solution concepts display the same phenomenon, so the clear implication is that we shall not succeed in finding a solution concept which is at the same time "satisfactory" in the sense of capturing our intuition about the effects of merger and "structure-free" in the sense that it can successfully be applied to any alignment of the players.

The examples collected in this paper are differentiated by the solution concept imposed and by their game-theoretic attributes.

For our purposes, the important types of games are differentiated according to:

1) whether or not utility is transferrable;

2) whether the set of players is finite, countable, a continuum or a mixed measure space;

3) whether the game is cooperative or noncooperative; and

4) whether the game can be derived from an underlying market situation.

The solution concepts we shall impose on these games include:

1) the core;

2) the Shapley value;

3) Nash, strong, and perfect equilibria;

4) competitive equilibrium; and

5) Nash Bargaining solution.

We have omitted from discussion such "structural" solution concepts as the von-Neumann-Morgenstern solution, the bargaining set family of solution concepts (including the kernel, the nucleolus, etc.), and solution concepts which specifically address the problem of coalition formation.

As far as possible, we have confined our attention to exampl· with direct economic relevance. However, some of the examples have much greater generality.

In a subsequent paper, (Cave and Salant (1980)) we pursue the second interpretation of these results, and construct a model of oligopoly where the industry structure is endogenous, and bears a close and clear relationship to industrial performance. This not only removes the problem raised by the phenomenon of "disadvantageous syndication, but also provides testable hypotheses which may aid in choosing between various solution concepts as descriptions of industry behavior.

I. Background and Literature

We begin by describing the disadvantageous syndicates phenomenon and noting examples that have already appeared in the literature. In games with more than two players, a group of players can consider the possibility of forming a syndicate or acting as a single player. In the context of a cooperative game, where all strategic considerations are omitted and each coalition C is represented by a (set of) utility(ies) it can obtain, the formation of a syndicate S simply removes from consideration those coalitions C which neither contain S nor are disjoint from S : whatever the members of S do, they do as a bloc. In strategic games, formation of S means that the members of S agree to adopt some *correlated* strategy, and possibly to carry out some side payments to redistribute their receipts. In a game with (non)transferrable utility being played according to a *point-valued* solution concept 1/ a syndicate is said to be disadvantageous if the solution obtained after the

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1/ E.g., Shapley value, Nash equilibrium, Nash bargaining solution.

syndicate forms gives less utility to (at least one member of) the syndicate. In a cooperative game played according to a set-valued solution concept 1/, the formation of a syndicate S usually enlarges the solution by removing some coalitions from consideration: we say that S is disadvantageous if the new allocations are worse for (at least some members of) S than those already in the solution.

We shall now mention some of the observations on this phenomenon that have found their way into the literature.

In respect of the core, the first examples of disadvantageous syndication were given by Aumann (1973) in the context of Shitovitz' (1973) model of a mixed Atomic/Nonatomic market with transferrable utility. Later, Samet (1971) showed that the phenomenon could not arise in case all the traders had differentiable utility functions, while Postlewaite and Rosenthal (1974) sought to overcome Aumann's example with an example where the disadvantages to syndication had an obvious validity.

S. Hart, (1974), working with the symetric von Neumann-Morgenstern solution in the context of a finite-type mixed market without transferrable utility found that each type must form a cartel, it being disadvantageous not to (given that all

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1/ Core, von Neuman-Morgenstern solution, Bargaining set, etc.

traders of the same type receive the same allocation 1/). Okuno, Pastlewaite, and Roberts (1980) found a similar phenomenon for Nash Equilibria of a Shapley-Shubik exchange game, and Salant, Switzer and Reynolds (1980) also produced an example of disadvantageous syndication in an oligopoly game under Nash Equilibrium.

In what follows we shall discuss the phenomenon of disadvantageous syndication. In this paper, we shall address: the noncooperative equilibrium concepts; the Nash Bargaining Solution; the Shapley value; and competitive equilibrium.

II. Noncooperative Equilibrium Concepts

2.1 Definition: A noncooperative game in normal form is a triple (N, Σ, ρ) where: N is the set of players (taken here to be finite); $\Sigma = \underset{i \in N}{\times} \Sigma^i$ are the strategies available to the players; and $\rho : \Sigma \to \mathbb{R}$ is the payoff function which measures the preferences of the players over outcome or strategy n-t in Σ.

2.2. Definition: $\sigma^* \in \Sigma$ is a Nash Equilibrium iff $\forall i$, $\forall \sigma^i \in \Sigma^i$, if $(\sigma^*(i), \sigma^i) = (\sigma^{*1}, \ldots, \sigma^{*i-1}, \sigma^i, \sigma^{*iH}, \sigma^{*n})$ we have

$$\rho^i(\sigma^*) \geq \rho^i(\sigma^*(i), \sigma^i)$$

σ^* is a strong Nash Equilibrium if $\forall\ C \subset N$, and $\forall\ \sigma^C \in \underset{i \in C}{\times} \Sigma^i \equiv \Sigma^C_u$ $\exists$ $i \in C$ s.c.

$$\rho^i(\sigma^*(C), \sigma^C) < \rho^i(\sigma^*)$$

1/ This "equal treatment" property, imposed by the symmetry of the VN-M solution, is not trivial; even the core fails to provide equal treatment in mixed markets.

(there are many variants of this definition) σ^* is a <u>perfect Nash Equilibrium</u> if the following is true: $\forall i$ let $\underset{\sim}{\varepsilon}^i$ be a set of weights on the pure strategies of i s. t. if $\Pi^i \subset \Sigma^i$ are the pure strategies $\int_{\Pi^i} \underline{\varepsilon}^i = 1$ and s.t. $\forall$ pure strategy π^i of i $(\sigma^{*i} + \underset{\sim}{\varepsilon}^i)(\pi^i) > 0$ i.e., $\sigma^{*i} + \underset{\sim}{\varepsilon}^i$ is a <u>completely mixed</u> strategy. Let $E^i(\sigma^{*i})$ be the set of all such perturbations: it is an open set. If we denote by $E^{(i)}(\sigma^{*(i)})$ the set $\underset{j \neq i}{X} E^j(\sigma^{*j})$ then σ^* is a perfect N.E. iff $\forall i$ and $\forall$ sequence $\varepsilon_t^{(i)} \to 0$ in $E^{(i)}(\sigma^{(i)})$, there is a sequence ε_τ^i with the properties

1) $\varepsilon_\tau^i \to 0$
2) $\forall \hat{\eta}^i \varepsilon E^i(\sigma^{*i})$

$$\rho^i(\sigma^{*i} + \varepsilon_\tau^i, \sigma^{*(i)} + \varepsilon_\tau^{(i)}) \geq \rho^i(\sigma^{*i} + \hat{\eta}^i, \sigma^{*(i)} + \varepsilon_\tau^{(i)})$$

2.3 <u>Remark</u>: The definition of <u>Nash Equilibrium</u> is of a situaation from which no individual can profitably and unilaterally defect. A strong Nash Equilibrium strengthens this concept by ruling out profitable multilateral defections by any coalition. Since the set of all players is itself a coalition, a strong equilibrium is <u>a fortiriori</u> Pareto optimal. The concept of a perfect equilibrium in the normal form is based on what Selten (1974) has called the "trembling hand principle". The assumption is that one's opponents may not actually be able to carry out their parts of the equilibrium, but may, with small but positive probabilities, make any move. If the stated strategy n-tuple is proof against such small mistakes, it is said to be perfect. There is a further qualification due to Myerson (1976), where the probability of mistake is proportional to the profitability of the mistake. Such equilibria are called proper. However, these refinements need not concern us, as our first example will demonstrate disadvantageous syndication in the context of a proper, hence perfect, hence Nash equilibrium.

2.4 Example: This is a noneconomic example: there are three players, each of whom has 2 moves. The game is displayed in bimatrix form below: the rows are indexed by the strategies T_1, B_1, of player 1 while the moves L_i, R_i of players $i = 2, 3$ are used to index the columns. In the entry corresponding to a particular choice of strategies, the payoff is shown. For example, if 1 plays T_1, 2 plays R_2 and 3 plays L_3, we find the triple (4,0,1) meaning that 1 gets \$4.00, 2 gets \$0.00 and 3 gets \$1.00. To avoid mixtures, we have so arranged matters that there is in each case a unique, perfect and proper equilibrium.

	L_3		R_3	
	L_2	R_2	L_2	R_2
T_1	(2,2,1)	(4,0,1)	(-1,1/2,0)	(1/2,-1,0)
B_1	(0,4,1)	(3,3,1)	(1/2,-1,0)	(0,0,2)

The unique equilibrium of this game is at (T_1, L_2, L_3) and gives a payoff of (2,2,1). If the syndicate (1,2) forms, it acts as a single player and chooses any convex combination of the four pure strategies indexing the rows of the following matrix in an attempt to maximize its total payoff.

move of {1,2}

	L_3	R_3
T_1 L_2	(4,1)	(-1/2,0)
T_1 R_2	(4,1)	(-1/2,0)
B_1 L_2	(4,1)	(-1/2,0)
B_1 R_2	(6,1)	(0,2)

This time the unique equilibrium is at (B_1, R_2, R_3) but the payoff to {1,2} is \$0 instead of the \$4 they get acting separately. What happened is that greater power lead to greater greed, which got them into trouble. To address the question of strong equilibria, notice first that, while there are no strong equilibria of the original game, the equilibrium of the syndicated game is also a strong equilibrium: any pareto optimal equilibrium of a 2-player game is strong by definition.

In a sense, this is an unsatisfactory way of dealing with the strong equilibrium since it might be objected that we have implicitly allowed transfers of utility in claiming that the syndicate acts to maximize its total payoff, while at the same time ruling out transfers in the definition of strong equilibrium. However, the difficulties are not so grave in this example, since the coalition {1,2} can achieve any transfers by correlation, without the necessity of involving sidepayments. As for the definition of strong equilibrium: if it were made on the basis of total payoff to the defecting coalition the number of games possessing strong equilibria would fall drastically. What we do need for use in such situations is some way to determine the syndicate's preferences from those of its members. This is a deep problem and we shall defer it to the second paper.

The other unsatisfactory aspect is that, in the original game, there were no strong equilibria. However, a little reflection will convince the reader that, in general situations, the formation of a syndicate strictly diminishes the set of coalitions whose strategic options must be taken into account when computing strong equilibria. Thus the set of strong equilibria enlarges and the appropriate notion of disadvantageous syndication is the set-valued one. To see how syndication may be disadvantageous in respect of strong equilibrium consider the following example.

2.5 Example: This is again a 3-player game with two pure moves for each player.

	r L_3		$(1-r)$ R_3	
	L_2 q	R_2 $(1-q)$	L_2 q	R_2 $(1-q)$
p T_1	(0, 0, 1)	(0, 0, 0)	(0, 0, 0)	(4, 4, 4)
$(1-p)$ B_1	(0, 1.1, 4)	(1, 1, 10)	(0, 0, 0)	(0, 0, 0)

This game has 2 pure equilibria:

(B_1, L_2, L_3) with payoff $(0, 1.1, 4)$

(T_1, R_2, R_3) with payoff $(4, 4, 4)$

in addition there are two kinds of mixed equilibria denoted by (ρ, q, r):

a) equilibria of the form $(p, 1, 1)$ with payoffs

$$(0, 1.1(1-p), r-3p)$$

b) equilibria of the form $(1, q, 1)$ with payoffs

$$(1.1(1-q), 0, 2-q)$$

(the equilibria of type a include the first pure equilibrium)

There is a single strong equilibrium at (T_1, R_2, R_3). Now suppose the coalition {1,2} forms. The game becomes

		L_3	R_3
p_1	T_1 L_2	(0, 1)	(0, 0)
p_2	B_1 L_2	(1.1, 4)	(0, 0)
p_3	T_1 R_2	(1.1, 2)	(8, 4)
$(1-p_1-p_2-p_3)$	B_1 R_2	(2, 10)	(0, 0)

Now there are but three equilibria:

(T_1, R_2, R_3) with payoff $(8, 4)$

(B_1, R_2, L_3) with payoff $(2, 10)$

and a mixed equilibrium at $p_1 = p_2 = 0$, $p_3 = .8333$ $q = .8988$ with payoff $(1.7977, 3.3333)$

We retain (T_1, R_2, R_3) as a strong equilibrium, giving payoffs of (8,4). However, there is now one additional strong equilibrium, at (B_1, R_2, L_3) with payoffs of (2,10). This shows that in this case, formation of the syndicate {1,2} is disadvantageous with respect to strong equilibrium.

III. Nash Bargaining Solution

Closely related to the notion of Nash equilibrium is Nash's solution to the bargaining problem with variable threats. To show that syndication is not necessarily an advantage in the context of this model, we shall begin by describing the fixed model.

3.1 Definition: A fixed-threat bargaining problem is a pair (A, d) where

$A \varepsilon \mathbb{R}^n$ is a set of possible agreements, represented as n-tuples of utility. It is taken to be compact and convex.

$d \varepsilon \mathbb{R}^\omega$ is the disagreement or threat point, representing the utility agents receive if they fail to agree.

For obvious reasons, we assume that

$$\{a \varepsilon A: \ a^i \geq d^i, \ a \neq d\} \neq \emptyset$$

(In other words, that there are some worthwhile agreements). This is the model appropriate to situations in which players must reach some agreement as to the division of the spoils before taking joint action. The solution to this problem is given as the unique function that maps problems into agreements and satisfies certain axioms. These are laid out in the Appendix.

For our purposes, it is sufficient to say that the Nash Solution maximizes the total gain from agreement. In other words, the Nash solution $\eta(A,d)$ is that agreement $a \varepsilon A$ for which the number

$$(a^1 - d^1)(a^2 - d^2) \; \ldots \; (a^N - a^N)$$

is the greatest, subject to the condition that $a^i \geq d^i$ for all i. In the fixed-threat model, due to the symmetry of the solution, all syndication is disadvantageous. Perhaps this claim can be clarified by means of an example.

3.2a Example: three-person, transferrable utility, fixed-threat bargaining. "Transferrable utility" in this context means that the set of possible agreements is part of a plane with slope -1: essentially, there is a certain amount of money to be divided between three players who all value it in the same linear (constant marginal utility) way. Suppose that the amount is \$1.00 and that the disagreement payoff is \$0. Then the Nash solution selects numbers a_1, a_2, and a_3 such that $a_1 + a_2 + a_3 = 1$, and such that the "Nash product" $a_1 a_2 a_3$ is maximised. It is easy to see that the allocation selected is $a_1 = a_2 = a_3 = 1/3$. If 1 and 2 form a coalition, we have a two-person bargaining problem, to which the Nash solution assigns a division of $b_1 = b_2 = 1/2$. If we divide the proceeds among the members of the coalition (1,2) using the Nash solution again, we get the three-person allocation for the "syndicated" game given by: $c_1 = c_2 = \frac{1}{4}$; $c_3 = \frac{1}{2}$, which is clearly worse for the members of the coalition (1,2).

In general games, without sidepayments, we would have trouble in describing syndication, since the data of the problem typically do not allow us to define the set of agreements available to subsets of the set of all players: the bargaining game is an unanimity game. However, we shall verify our claim for two situations where we <u>can</u> describe syndication.

3.2b <u>Theorem</u>: Let (A,d) be a transferrable utility bargaining problem (i.e., the Pareto-optimal set is an hyperplane). Then $\forall\ C \quad N$, if (A_C, d_C) is the bargaining problem when C forms [$\#C \geq 2$]:

$$\sum_{i \in C} \eta^i(A,d) > \eta^C(A_C, d_C)$$

<u>proof</u>: First we must describe (A_C, d_C)

Let $\Psi_C: \mathbb{R}^n \to \mathbb{R}^{n-\#C+1}$ be $\Psi_C(x^1, \ldots, x^n) = (\Psi_C^j(x), j\ c, \Psi_C^C(x))$

$$\Psi_C^j(x^1, \ldots, x^n) = x^k \quad \text{for } j \notin C$$

$$\Psi_C^C(x^1, \ldots, x^n) = \sum_{i \varepsilon C} x^i$$

$$\text{then } A_C = \Psi^C(A)$$

$$d_C = \Psi^C(d)$$

and the content of the theorem is that if

$$\tilde{a} = \arg\max_{\varepsilon A_c} \prod_{j \varepsilon c} (a^j - d_c^j)(a^c - d^c),$$

$$\text{i.e., } a = \arg\max_{a \varepsilon A} \prod_{j \notin c} (a^j - d^j)(\sum_{i \varepsilon c} a^i - d^i)$$

then $\sum_{i \varepsilon c} \tilde{a}^i$ does <u>not</u> maximize $\prod_{i \varepsilon c} (a^i - d^i)$

To simplify the argument, we first write

$$B = A - d \quad B_c = A_c - d_c$$

so that the original problem is then

$$\text{I} \quad \max_{b \varepsilon B} \prod_{j \varepsilon N} b^j$$

and the problem after syndication is

$$\text{II} \quad \max_{b \varepsilon B} (\prod_{j \varepsilon c} b^j)(\sum_{i \varepsilon c} b^i)$$

To further simplify matters we note that the constraint $b \varepsilon B$ can in this case be written as

$$\sum_{j \varepsilon n} \lambda^j b^j = k$$

where $\lambda^j > 0$ (all j) are fixed weights. The solution to problem I under these conditions can be found from the first-order for maximizing

$$\delta_I = \prod_{j\in N} b^j + \mu(k - \sum_j \lambda^j b^j)$$

i.e., (neglecting corner solutions)

$$\forall_j \quad \prod_{k\neq j} b^k - \mu\lambda^j = 0 \quad \text{or} \quad \mu\lambda^j b^j = \prod_{k\in N} bk \quad \forall_j$$

thus $\lambda^j b^j$ is independent of j and by feasibility

$$\sum_j \lambda^j b^j = \frac{\prod_k b^k}{\mu} = k = n\lambda^j b^j$$

so $\forall_j$

$$b^j = \frac{k}{n\lambda^j}$$

and $\sum_{i\in c} \lambda^i b^i = \frac{k\#c}{n}$

Before turning to problem II, we remark that, by virtue of the affine invariance of the solution we can write $\lambda^j = 1$. The first order conditions for II are now

$$j\epsilon C(\prod_{k\notin c\cup j} b^k)(b^c) - \mu = 0$$

$$C: \quad (\prod_{j\notin c} b^j) - \mu = 0$$

$$\therefore \quad b^j = \frac{k}{n-c+1} = b^c$$

To see the conclusion of the theorem compare

$$\frac{kc}{n} \quad \text{and} \quad \frac{k}{n-c+1}$$

so syndication is disadvantageous iff

$$\frac{kc}{n} > \frac{k}{n-c+1}$$

i.e., $nc - c^2 + c > n$

i.e., $n(c-1) > c(c-1)$

i.e., $n > c > 1$. QED.

Instead of proving the theorem for the more general case, it may be instructive to consider an example of bargaining with out transfers of utility.

3.3 Example: Let a certain amount of money $\$k$, be specified and suppose there are n agents trying to divide this money. Suppose that agent i's utility for money is given by

$$\mu^i(x) = \gamma_i \ln(x)$$

where $\gamma^i \varepsilon (0,1)$. To further simplify the example, we assume that $k > n$ and that if the agents fail to agree they each get $\$1$ for trying. First, we can easily solve the n-person problem, upon noting that in all such money division problems with reference levels of 0 the first-order condition for the Nash solution is

$$\frac{U^i(x^i)}{MU^i(x^i)} = \text{const.}$$

Where $MU^i = \dfrac{du^i}{dx^i}$ is i's marginal utility. This translates to

$$x_i \ln x_i = \text{const.}$$

but since $x \ln x$ is a monatone increasing function of x, this means that the Nash solution calls for equal division.

$$x^i = x^j = \frac{k}{n}$$

Now consider the effect of syndication: a group C forms and agrees to divide any joint proceeds according to the bargaining solution. In other words, if C gets by concerted action an amount of money X^C they will then bargain over this. Thus, within C we have again the equality of $\frac{U^i}{MU^i}$. This gives C as virtual utility function W^C which reflects the total utility numbers of C get by acting in concert:

$$W^C(x^C) = \sum_{i \varepsilon C} U^i(x_i^C)$$

where x_i^C is the bargaining solutions division of x^C among the $i \varepsilon C$. In fact in this case the amount x^C will be equally divided, by the previous argument, so if $c = \#C$

$$x_i^C = \frac{x^C}{c}$$

$$W^C(x^C) = \sum_{i \varepsilon C} u^i(x_i^C) = \sum_{i \varepsilon C} \gamma_i \; [\ln x^C - \ln c]$$

and

$$-\frac{W^C(x^C)}{MW^C(x^C)} = x^C[\ln x^C - \ln c]$$

Thus, the Nash solution for this two-stage procedure satisfies the following conditions, where $\tilde{x} = (\tilde{x}^j : j \varepsilon c, \tilde{x}^C)$ is the allocation given by the solution:

1) by symmetry $\tilde{x}^j = \tilde{x}^{j'} \equiv \tilde{x}^{1'}$ $\forall j, j' \not\in C$

$$[\text{i.e., } \frac{U^j}{MU^j} = x^j \ln x^j = x^{j'} \ln x^{j'} = \frac{U^{j'}}{MU^{j'}}$$

2) $(n-c)\ \tilde{x}' + \tilde{x}^c = k$ (feasibility)

3) $\tilde{x}' \ln \tilde{x}' = x^c\ [\ln \tilde{x}^c - \ln c]$ (Nash condition)

It is easy not to show the disadvantages of syndication:

In the range $\tilde{x}' \geq 1$, $\tilde{x}' \ln \tilde{x}' \geq 0$. But if $\tilde{x}^c[\ln \tilde{x}^c - \ln c]$ ≥ 0 and $\tilde{x}^c > 0$, then $\tilde{x}^c\ [\ln \tilde{x}^c - \ln c]$ is monotone increasingly, as is $\tilde{x}^1 \ln \tilde{x}^1$. Now try $\tilde{x}' = \frac{k}{n}$, $\tilde{x}^c = \frac{ck}{n}$ as before: $\tilde{x}^c = c\tilde{x}'$ so condition 3 becomes

$$\tilde{x}' \ln \tilde{x} = C\tilde{x}' \ln \tilde{x}'$$

Thus $\tilde{x}^c[\ln \tilde{x}^c + \ln c]$ $\tilde{x}' \ln \tilde{x}'$ and so by monotonicity we must reduce $\tilde{x}^c$ (while simultaneously increasing $\tilde{x}'$ by condition 2 above). Therefore, the syndicate winds up with less money, and so each member has lower utility. We can prove a generalized version of Theorem 3.2 appropriate to situations of money division, but the proof is not very enlightening.

In the above model, one highly counterintutive feature is the existance of a fixed threat with respect to which a strong measure of symmetry exists: either side can invoke the threat by refusing to agree, and this is all either side can do. In real bargaining situations, however, there are usually a range of possible threats and counter threats. The bargaining solution has been modified to take this into account, albeit in a highly specialized way.

3.4 Definition: A variable threat bargaining game is (A, Σ, ρ) where A is the set of possible agreements subject to the same assumptions as before; and (Σ, ρ) is a game in normal form (see def 2.1) called the threat game.

This game is played as follows: each n-tuple of threat strategies $\sigma \in \Sigma$ leads to a possible threat payoff $\rho(\sigma) \in \mathbb{R}^n$. If we denote by $h(A, d) \in \mathbb{R}^n$ the Nash bargaining solution to the fixed-threat problem (A,d), we can extend this to give a new payoff function to the threat game:

$$H_A : \Sigma \to \mathbb{R}^n \quad \text{defined by}$$
$$H_A : (\sigma) = h(A, \rho(\sigma))$$

a Nash equilibrium (not solution) of this new noncooperative game (Σ, H_A) is called a variable-threat bargaining solution, and the strategies $\sigma^* \in \Sigma$ used at this equilibrium are called optimal threats. For our purposes the fact that both the fixed-threat bargaining solution and Nash equilibrium are used to define the variable-threat solution is ample evidence that disadvantageous syndication can and does occur in this context as well.

IV. The Shapley Value

In this section, we turn to what are called cooperative games, or games in characteristic function form. In passing from the extensive (tree) form of a game, where the order in which players move is displayed, to the normal or strategic form, we obliterated certain strategic considerations. Now we are going to altogether remove any considerations of individual strategy in order to focus on the coalitional aspects of the game. As we shall show, it is possible to obtain several characteristic functions from strategic or normal form games, but the reverse passage can only be made in a trivial manner. In keeping with most of the rest of this paper, we shall stick to games of transferrable utility which are a special case of games without transferrable utility.

4.1 Definition: Let (T, ℓ, μ) be a probability space. T is a (possible finite) set of players. ℓ is a σ-field (see appendix) of subsets of T called coalitions, and $\mu:\ell\to[0,1]$ is a $(\sigma-)$ additive measure with $\mu(T)=1$. For the reader unfamiliar with these concepts, it is possible to think of

$$T = \{1, \ldots n\}$$

$$\ell = \{S \subseteq T\}$$

$$\mu(S) = \frac{\#S}{n}$$

a game is a function $\nu:\ell \to \mathbb{R}$ s.t. $\nu\emptyset) = 0$. The game is monotonic if $S \quad S' \Rightarrow \nu(S) \geq \nu(S')$; and superadditive if $\forall S, S'\epsilon\ell$ s.t. $S \quad S' - \emptyset$, $\nu(S) + \nu(S') \leq \nu(S \quad S')$.

4.2 Remarks: The number $\nu(S)$ is called the worth of the coalition S, and reflects the total amount of money, utility, etc. that S has to distribute among its members. From the data given by ν, cooperative game theory seeks to predict what allocations will be made to the individual players and under some circumstances, what coalitions will form. Such games are quite common: two examples are

i) 2/3 majority rule

$$(S) = \begin{cases} \text{if } \mu(S) \geq 2/3 \\ \text{otherwise} \end{cases}$$

ii) monetary exchange economy: each trader τ has an endowment $\underset{\sim}{e}_\tau \in \mathbb{R}_t$ and a utility function $u_\tau : \mathbb{R}^\ell_t \to \mathbb{R}$.

$$\nu(S) = \sup \{\int_S u_t(\underset{\sim}{x}_t)\, \mu(dt) : \int_S \underset{\sim}{x}_t \mu(dt) = \int_S \underset{\sim}{e}_t \mu(dt)\}$$

that is, each coalition can redistribute its resources in any way it likes to promote the greatest total utility.

In example ii we see the presence of some strategic considerations: actual redistributions of goods are involved. In general, given any normal form game (N,Σ,ρ) we can derive two characteristic functions V_α and V_β as follows:

i) $V_\alpha(S) = \max \{\nu : \exists\, \sigma^s \in \underset{i \in S}{X}\, \Sigma^i$ s.t.

$$\forall\, \tilde{\tau}^{(s)} \in \underset{j \notin S}{X}\, \Sigma^j ,\ \nu \leq \sum_{i \in S} \rho^i\, (\sigma^s, \tau^{(s)})\}$$

ii) $V_\beta(S) = \max \{\nu : \forall\, \tilde{\tau}^{(s)} \in \Sigma^{(s)},\ \exists\, \sigma^s \in \Sigma^s$ s.t.

$$\nu \leq \sum_{i \in S} \rho^i\, (\sigma^s, \tilde{\tau}^{(s)})\}$$

Thus, the <u>α-worth of S</u> is the amount that s can guarantee if it has to move first (is not allowed to react to T/S's specific choice), while the β-worth of S reflects the amount s can guarantee if they can react to T/S with a specific defense. Clearly, $V_\alpha(s) \leq V_\beta(s)$. At any rate, there are many ways of solving such cooperative games, and of constructing them to model features of the real economy. Before looking at syndication in the context of any specific examples, we must define it.

4.3 <u>Definition</u>: Let (T, ℓ, μ), ν be a game, and $S \varepsilon \ell$. The <u>game with syndicate</u> S, or V_s is the game formed by restricting the range of ν to coalitions S' that do not "break up S" i.e.,

$$V_s : \ell_s \to \mathbb{R}$$

$$\text{where } \ell_s = \{S' : S' \cap S = \emptyset \text{ or } S' \cap S = S\}$$
$$\forall\, S' \varepsilon \ell_s\ ,\ V_s(S') = \nu(S')$$

If we have to deal with several disjoint coalitions, we can describe them by means of a <u>partition</u> π of T :

$$\pi \subset \ell \text{ which satisfies } \cup\{S' \ \varepsilon\ \pi\} = \ell$$
$$\forall\, S', S'' \ \varepsilon\ \pi,\ S' \cap S'' = \emptyset$$

then <u>the game relative to the partition</u> π, V_π is the restriction of ν to-

$$\ell_\pi = \{S' \varepsilon\ \ell : \forall S'' \varepsilon \pi \text{ either } S'' \cap S' = \emptyset \text{ or } S'' \cap S' = S''\}$$

In other words, our view of syndication is that the syndicate members act as an indissoluble unit. Among the ways of solving cooperative games, the Shapley value is one of the most thoroughly explored. It has an axiomatic characterization as the unique solution with certain efficiency and equity properties (see appendix) as well as probabilistic and constructive characterizations. It was expressly designed to aid in the analysis of *power*, and has been used for analyses of political and economic power, allocation of inframarginal costs, allocation of school funds, etc. The axioms that give rise to the value are essentially the same as those giving the Nash bargaining solution in a different context.

The value gives to each player an amount that may be described as the player's "expected contribution to a random coalition. To indicate why we might expect syndication to be advantageous in the context of the value, we quote from Aumann's discussion of his example of disadvantageous syndication in the core:

> "The concept of core is based on what a coalition can guarantee for itself. Monopoly power is probably not based on this at all, but rather on what the monopolist can *prevent* other coalitions from getting. His strength lies in his threat possibilities, in the bargaining power engendered by the harm he can cause by refusing to trade. Put differently, the monopolist's power - and for that matter, that of any other trader - is measured by the difference between what others can get with him and what they can get without him. This line of

> reasoning is entirely different from that used in the definition of the core. But it is not foreign to game theory; indeed it is closely related to the ideas underlying the Shapley value."

We begin by writing out the formulae for the value of a three-person game: there are six possible orders of the players, and if each order has equal probability (this is how the random coalitions are generated) we get, writing $\Psi v(i)$ for the value of player i

$$\Psi v(1) = \frac{1}{6}(2v(1) - v(2) - v(3) - 2v(2,3) + v(1,2) + v(1,3) + 2v(1,2,3))$$

$$\Psi v(2) = \frac{1}{6}(2v(2) - v(1) - v(3) - 2v(1,3) + v(1,2) + v(2,3) + 2v(1,2,3))$$

$$\Psi v(3) = \frac{1}{6}(2v(3) - v(1) - v(2) - 2v(1,2) + v(1,3) + v(2,3) + 2v(1,2,3))$$

It will be noted that the total payoff, $\Psi v(1) + \Psi v(2) + \Psi v(3) = v(1,2,3)$, so the value is <u>efficient</u>; based on the assumption that the all-player coalition can guarantee the biggest total, the value is a scheme for dividing this total. The total amount earned by the coalition (1,2) under this scheme is readily seen to be:

$$\Psi v(1) + \Psi v(2) = \frac{1}{6}(v(1) + v(2) - 2v(3) - v(2,3) + 2v(1,2) + 4v(1,2,3))$$

On the other hand, we could treat them as a single player: in other words, we would have a two player game in which the only coalitions that could form are those that either contain or are disjoint from the coalescence (1,2). In this case, there are only two orders of the players, and the Shapley values are:

$$\Psi v(1,2) = 1/2(v(1,2) + v(1,2,3) - v(3))$$

$$\Psi v(3) = 1/2(v(3) + v(1,2,3) - v(1,2))$$

From these data, it is easy to see that players 1 and 2 benefit from merging if and only if

$$v(1) + v(2) + v(3) + v(1,2,3) < v(1,2) + v(2,3) + v(1,3)$$

and the symmetry of this condition makes it clear that this is true for any other coalition we might consider.

We now present economic examples of both profitable and unprofitable merger. Both have quasi-concave utilities and reasonable endowments (from the monopoly-theoretic point of view), so neither is pathological.

4.4 Example: We begin with a 3-agent 3-good economy, where the endowments are given by:

Player	Endowment of	x	y	z
1		1	0	0
2		0	1	0
3		0	0	1

a) Profitable Merger: $U_i(x,y,z) = xy + yz + xz - xyz$. Using the formula for $\nu(S)$ given in 4.2 iii above, we get the following, representing the best each coalition can do with its endowment:

$$V(i) = U(e_i) = 0 \qquad i = 1,2,3$$

$$V(i,j) = U(e_i + e_j) = 1 \qquad \text{for } i \neq j$$

$$V(1,2,3) = U(e_1 + e_2 + e_3) = 2$$

[The reader can easily check that with these utility functions, the best total utility is achieved by giving all the goods to one agent]. This characteristic function satisfies the condition for profitable merger.

b) unprofitable merger: $U_i(x,y,z) = xy + yz + xz + xyz$

$$V(i) = U(e_i) = 0$$

$$V(i,j) = U(e_i + e_j) = 1$$

$$V(1,2,3) = U(e_1 + e_2 + e_3) = 4$$

We now turn to another cooperative solution concept.

V. The Core

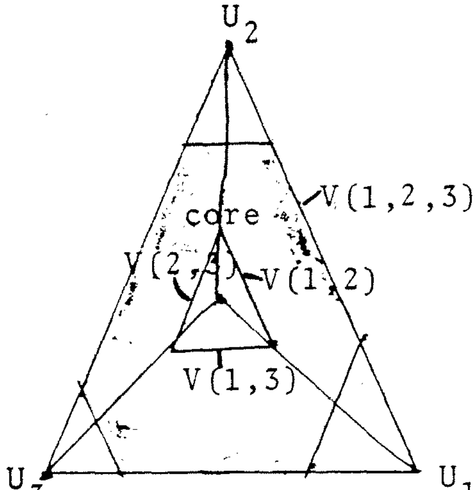
A great deal has been written about the Core in economics, so we shall not repeat the discussion here. Suffice it to say that an allocation $\underset{\sim}{x} : T \to \mathbb{R}$ of utility or money is in the core iff, $\forall\, S \in \ell$

$$\int_S x_t \;\mu(dt) \geq \nu(S) \;.$$

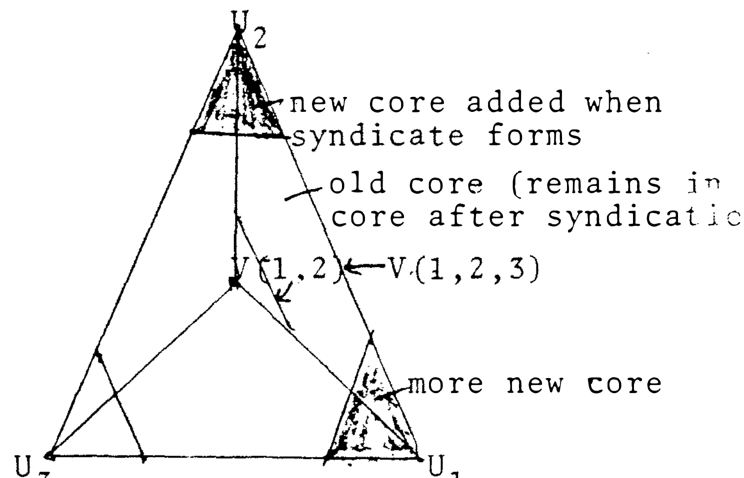
In Aumann's example 3, we were presented with a situation in which, given that the monopoly formed as a single entity (atom), the core consisted of two points: a competitive allocation which would result if the monopoly acted as independent units, and one other allocation which required the exercise of monopoly power. The payoff to the monopoly at the latter allocation was strictly worse than at the competitive allocation. We will merely present the cores derived from one of the games of example 4.4, in graphic form.

5.1 Example

Transferrable-utility core: The following is a diagram in utility space of the game of example 4.4b: The lines connecting pairs of axes are the possible redistribution of the worth of that two-person coalition to its members: the plane represents redistributions of V(1,2,3). The core is the shaded area: that part of the V(1,2,3) surface lying above all the V(i,j) lines when the latter are projected out. Next to it we have drawn the same figure for the syndicated case, and have shaded only the portions added to the core. It will be noted that neither shaded region represents a gain for <u>both</u> members of the coalition.



before syndication



after syndication

This is a rather weak notion of disadvantageous syndication, but we can easily construct stronger examples in a non-transferrable-utility context, where the $\nu(S)$ curves may have proper curvature. Full discussion of this situation will be postponed to a subsequent paper, but we present below a simple non economic example: the functions $\tilde{\nu}(S)$ are now given as <u>sets of utilities</u> - the transferrable-utility $\nu(S)$ is now $\tilde{\nu}(S) = \{\underset{\sim}{u}: T \to \mathbb{R}_t : \int_S u_t \mu(dt) = \nu(S)\}$.

5.2 Example: $T = \{1,2,3\}$

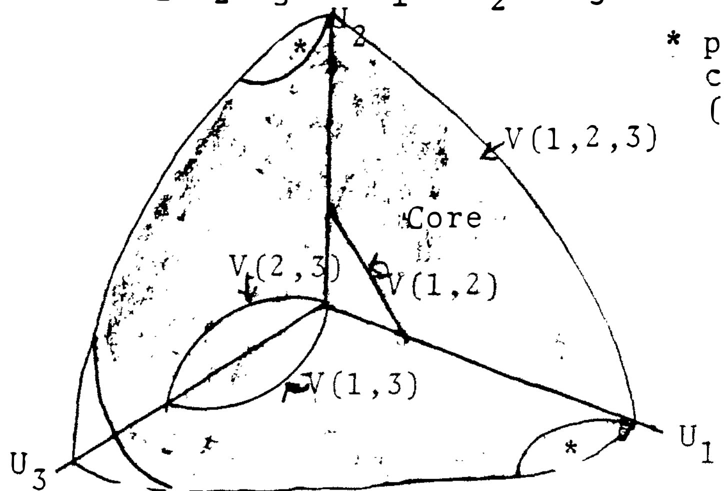
$\nu(i) = 0 \ \forall i$

$\nu(1,2) = \{(U_1,U_2,U_3): \ U_1 + U_2 \leq 1\}$

$\nu(2,3) = \ (U_1,U_2,U_3): \ U_1 \leq 1 - 4(U_3 - 1/2)^2: \ U_3 \leq 1\}$

$V(1,3) = \{(U_1,U_2,U_3): \; U_1 \leq 1 - 4(U_3 - 1/2)^2: \; U_3 \leq 1\}$

$V(1,2,3) = \; (U_1,U_2,U_3): \; U_1^2 + U_2^2 + U_3^2 = 10\}$



* portions added to the core when the syndicate (1,2) forms

It is clear from the above diagram that syndication only introduces inferior outcomes into the core.

VI. Competitive Equilibrium

The competitive equilibrium is not really a game-theoretic concept, since it obviates any notion of market power, but we shall calculate a simple example to show that syndication is disadvantageous in this context as well.

6.1 Example: As before, we have a three-agent, three-good economy. This time, however, we use utility functions that are explicitly of the transferrable utility type: the third good (money) enters each agent's utility function linearly. In the previous example, the linearity in z was achieved only by symmetry, which ensured that the marginal utility of an extra unit of z was the same constant: in those cases, maximising the total utility of a coalition involved giving all the goods to a single member, whereas in the present case there is a nontrivial optimal division.

The endowments are as before: agent 1 has $e_1 = (1,0,0)$; $e_2 = (0,1,0)$; and $e_3 = (0,0,1)$. Agent i's utility function is:

$$u_i(x,y,z) = a_i \ln(x) + (1-a_i)\ln(y) + z$$

where the a_i are positive real numbers between 0 and 1; we assume that the sum $a_1 + a_2 + a_3$ is strictly between 1 and 2 for case of exposition. Given a vector $(p_x, p_y, 1)$ of prices, we can calculate <u>demand functions</u> in the usual way: each agent i solves:

$$\max\ u_i(x,y,z) \text{ subject to } p_x x + p_y y + z \leq p_x e_i^x + p_y e_i^y + e_i^z$$

The results of these calculations, which are summarized below, can then be used to compute the equilibrium prices and allocations; find the price at which demand equals supply, and the amounts demanded at those 'clearing prices'.

agent	demand for x	demand for y	demand for z
1	a_1/p_x	$(1-a_1)/p_y$	p_x-1
2	a_2/p_x	$(1-a_2)/p_y$	p_y-1
3	a_3/p_x	$(1-a_3)/p_y$	0

For this economy, which we shall think of as the unmerged case, the equilibrium is described as follows (letting $b = a_1+a_2+a_3$):
$p_x^* = b$; $p_y^* = 3-b$

agent	allocation of x	allocation of y	allocation of z
1	a_1/b	$(1-a_1)/(3-b)$	$b-1$
2	a_2/b	$(1-a_2)/(3-b)$	$2-b$
3	a_3/b	$(1-a_3)/(3-b)$	0

For our purposes, it is also important to know the combined utility of agents 1 and 2.

$$u_1^* + u_2^* = a_1\ln(a_1) + a_2\ln(a_2) + (1-a_1)\ln(1-a_1)$$

$$+ (1-a_2)\ln(1-a_2) - (a_1+a_2)\ln(b) - (2-a_1-a_2)\ln(3-b)$$

If agents 1 and 2 form a coalition, we shall assume that they will act as a single agent using the best possible redistribution of the goods between themselves to provide a virtual utility function. In particular, if agents 1 and 2 had a fixed amount (x,y,z) to divide between themselves in such a way as to maximize the sum of their utilities, it is easy to see that the allocation which does the job is (keeping the same column headings as above):

1	$(a_1x)/(a_1+a_2)$	$((1-a_1)y)/(2-a_1-a_2)$	any
2	$(a_2x)/(a_1+a_2)$	$((1-a_2)y)/(2-a_1-a_2$	any

Consequently, if we let $c = a_1+a_2$, we obtain the following demand function for the 'agent' (1,2): this agent has an endowment of (1,1,0) and a virtual utility function given by:

$$u_{12}(x,y,z) = d_1 + d_2 + c\ln(x) + (2-c)\ln(y) + z$$

where $d_1 = a_1\ln(a_1) + a_2\ln(a_2) - \ln(c)$, and $d_2 = (1-a_1)\ln(1-a_1)$ -

$+ (1-a_2)\ln(1-a_2) - (2-c)\ln(2-c)$

are constants. The demand function are as follows:

$$x^*_{12} = (c)(p_x + p_y - 1)/(2p_x)$$

$$y^*_{12} = (2-c)(p_x + p_y - 1)/(2p_y)$$

$$z^*_{12} = 1$$

Since this is a two-agent economy, the equilibrium prices are determined only up to the ratio of p_x to p_y. Thus we may set $p^*_x = 1$ as well. In this equilibrium, $p^*_y = 2(1-a_3)/c$. Using the optimal split derived above, we can write the allocation corresponding to this equilbrium as:

agent	allocation of x	allocation of y	allocation of z
1	$a_1(1-a_3)/c$	$(1-a_1)/2$	z (any z between 0 and 1)
2	$a_2(1-a_3)/c$	$(1-a_2)/2$	1-z
3	a_3	c/2	0

In this case, the total utility of the two merged agents is given as

$$u^*_{12} = d_1 + d_2 + c\ln(1-a_3) + (2-c)\ln((2-c)/2) + 1$$

After some manipulations, we can derive the change in utility upon forming the merged entity:

$$u^*_{12} - u^*_1 - u^*_2 = c(\ln(c)+\ln(b)+\ln(1-a_3)+\ln(3-b)+\ln(2)) - 2(\ln(3-b) + \ln(2))$$

This formula may be rather opaque, but we can see immediately that it can assume either sign if a_3 is close to 1 and both a_1 and a_2 are close to 0, the two large negative terms $\ln(c)$ and $\ln(1-a_3)$ will dominate everything else and ensure the unprofitability of mergers. On the other hand, if δ is a very small positive number, and we set $a_1 = a_2 = 1 - \delta$, while $a_3 = \delta$, the difference becomes $2(1-\delta)(2\ln(2) + 2\ln(1-\delta) + \ln(1+\delta) + \ln(2-\delta) - 2\ln(2(1+\delta))$. As δ becomes arbitrarily small, this tends to $6\ln(2)$, which is certainly positive, indicating the profitability of merger.

In our next note, we shall take up the issues of nontransferrable utility, market power, and some more sophisticated solution concepts.

Appendix: Axiomatic basis of Nash Solution and Shapley Value

I. Nash Solution: a bargaining game is a pair (A,d) where A is the set of agreement points and d is the disagreement or threat point; both objects live in utility space. It is convenient to assume that d belongs to A and that A is compact and convex. The axioms do not require selection of a single point, so we shall state them in terms of a set S(A,d) that depends on the data of the problem.

Axiom I: S(A,d) A: the result should be feasible;

Axiom II: if $y \in A$ is Pareto-dominated by some other point $x \in A$ (that is, x is better for everyone than y) then $y \notin S(A,d)$: the result should be Pareto-optimal;

Axiom III: if $x \in S(A,d)$, then $x_i \geq d_i$: the result should be individually rational;

Axiom IV: if A and B are two possible agreement sets, with $d \in A$ B, and if $x \in A$ and $x \in S(A,d)$ - if x was chosen when a larger set of agreements was available, then it should be chosen from a smaller set - the technical term for this axiom is Independence of Irrelevant Alternatives;

Axiom V: if A and d are both symmetric, in the sense that any interchange of the names of the players gives us the same problem back again - in particular, this requires that each player get the same disagreement payoff - then the result should be symmetric;

Axiom VI: the solution is not affected by (independant) linear transformations of utility scales - this is called linearity.

The theorem is that there is a unique function satisfying these six axioms; it is what we have called the Nash bargaining solution.

II. Shapley value: a game in characteristic function form is a a pair (N,v) where N is the set of players and v is a function that associates a number $v(S)$ to each subject S of N, with the added convention that $v(\emptyset) = 0$. A value is an operator that associates to each game a vector of payoffs, one to each player in the game. We denote this vector for a particular gave v by

$$\Psi v = (\Psi v(1), \Psi v(2), \ldots, \Psi v(n))$$

and ask that it satisfy the following axioms:

Axiom I: (symmetry) if i and j are two players with the property that, [if $T$is any coalition containing neither i nor j then $v(T\ i) = v(T\ j)$], then $\Psi v(i) = \Psi v(j)$ - if i and j are substitutes in the game, they should get the same thing;

Axiom II: (<u>null player</u>) if i is a player with the property that $v(T \quad i) = v(T)$ for all $T \quad N$, then $\Psi v(i) = 0$ - if i never adds anything, it should not get anything;

Axiom III: (<u>efficiency</u>) $\sum_{i=1}^{n} \Psi v(i) = v(N)$

Axiom IV: (<u>additivity</u>) let u and v be games with the same set of players, and define a new game $w(= u + v)$ by:

$$w(S) = u(S) + v(S)$$

Then $\Psi w(i) = \Psi u(i)$ for all players $i \quad N$.

The Shapley value is the unique operator satisfying these conditions.

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