WORKING
PAPERS

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WORKING PAPER NO. 39

May 1980

BUREAU OF ECONOMICS
FEDERAL TRADE COMMISSION
WASHINGTON, DC 20580

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* This article is based on work in my Ph.D. dissertation done at Northwestern University. I acknowledge the great aid given to me by my dissertation advisor, John Roberts.

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I. Introduction

One of the first oligopoly models taught to an economics student is the well-known Cournot model in Cournot (1838), and even a casual glance at the literature reveals this model is still receiving serious attention.[1] In this model, we assume that the firms choose the quantity they are to produce, and the prices from each firm are to be determined by the market so that markets clear. The predicted outcome is the Cournot equilibrium, a selection of quantities from all firms where there is no incentive for any firm to change its quantity unilaterally. Thus, in a market with n firms producing a homogeneous product, with an invertible market demand function D(p), and with a cost function $C_i(q_i)$ for each firm i = 1,2,...,n, a Cournot equilibrium is a $q^* \equiv (q_1^*,\ldots,q_n^*)$ where for all i

$$D^{-1}\Big(\sum_j q_j^*\Big)\, q_i^* - C_i(q_i^*) = \max_{q_i \geq 0} D^{-1}\Big(\sum_{j\neq i} q_j^* + q_i\Big) q_i - C_i(q_i)$$

$$= \max_{p_i} p_i\Big(D(p_i) - \sum_{j\neq i} q_i^*\Big) - C_i\Big(D(p_i) - \sum_{j\neq i} q_i^*\Big)$$

$$\text{subject to } D(p_i) - \sum_{j\neq i} q_i^* \geq 0 .$$

Two early critiques of the Cournot model are invariably included in any presentation of this model to the uninitiated. The first is in Bertrand (1883), where Bertrand argues that since each firm controls its price an appropriate model should where the feasible choices for each firm are identical to those choices actually available to a firm in a real market. For this model we will use a game theoretic structure, as it allows us to separate the analysis of feasible choices from the analysis of good choices. In Section II we construct this game for an oligopoly market with homogenous goods. In Section III, after this game is constructed, we examine those outcomes which are associated with Nash equilibria and compare these outcomes to those predicted by Cournot, Bertrand, and Edgeworth. Here, we find that no equilibrium exists in typical oligopolistic environments satisfying standard assumptions except that each firm has the ability to choose both the price to charge for its goods and the quantity of these goods to offer for sale -- a generalization of Edgeworth's result. In particular, this means the outcomes predicted by Cournot and Bertrand are both irrational. While it would take empirical evidence (of which there is plenty) to reject the oligopolistic outcomes predicted by Cournot and Bertrand, the result presented here indicates that, at the minimum, the story given to justify the use of either of these predicted outcomes is inappropriate. Even if either of these predicted outcomes is a satisfactory predictor, we would be right but for the wrong reasons. In Section IV we extend this model to oligopoly markets with differentiated products and examine the equilibrium outcomes. Here, we find again that no equilibrium exists in typical oligopolistic environments with differentiated products, environments satisfying firms are the players, and any choice of a nonnegative price and nonnegative quantity is a feasible physical action for each firm.

In addition to the physical action another part of defining a strategy of a game, a part often left unspecified, is the information available to each decisionmaker before a decision must be made. The standard information assumption in game theory is complete information, where it is assumed each player knows all of the payoff functions, which means knowing the set of feasible strategies for each player plus the "utils" received by each player from any selection of strategies. Note that, even though any selection of strategies determines some physical outcome and this physical outcome determines the payoffs, each player is not required to know the physical outcome by the complete information assumption. This allows a game where the perceived payoff functions yield some payoffs for some selections of strategies which are actually unattainable.[3] However, if it is reasonable to assume that each firm has the knowledge to satisfy the complete information requirement, then it is also reasonable to assume that each firm knows the intermediate physical outcome and, so that this information is not self-contradictory, this perceived physical outcome is attainable. I believe that this extra informational requirement, while not formally included before, is expected when one assumes each firm has full knowledge of the market and its participants. Thus, we assume the game we construct satisfies the complete equals the amount offered to the market; and if a surplus results the rationing mechanism determines the constraints to be imposed on some sellers so that ultimately the amount sold equals the amount demanded. There are, however, many different rationing mechanisms that could be used to assure a feasible trade when the amount demanded does not equal the quantity offered to the market. With a shortage, for example, different rationing mechanisms may force different consumers to sacrifice, yielding different responses for substituting other goods, so that the outcomes for these other goods may be different with different rationing mechanisms. We see that to determine the amount sold by each firm when markets do not clear, a specific rationing mechanism must be given to determine the constraints to be imposed on each individual and thus their choices. Nevertheless, when considering the effect of rationing mechanisms we do not wish to limit our examinations to any particular rationing mechanism, since many different ones are actually used in different circumstances and then only implicitly. Because of this, the results developed here are phrased to hold for any of a large class of rationing mechanisms, those yielding demand functions for the individual firms which are said to be consistent with a given market demand function.

The individual demand functions (D_i) are defined to be consistent with a given market demand function D if for every $s \equiv (p_1,q_1,\ldots,p_n,q_n)$, where $p \equiv \min\{p_i\}$ and $M \equiv \{i:p_i=p\}$,

(i) $D_i(s) = D(p) - \sum_{\substack{j \in M \\ j \neq i}} \min\{D_j(s),q_j\}$ for all $i \in M$;

(iii) If the firms offering the lowest price do not meet the market demand, some consumers may choose to purchase some of the good from a firm offering a higher price. In this case, some consumer must be rationed, leaving him with a decision problem that may be illustrated as in Figure 1. Here, consumer ℓ may purchase some amount of good X from those firms offering the lowest price p, some of good X from firm i at the price $p_i > p$, and some of good Y at the price p_y. Consumer ℓ is limited by the rationing mechanism from purchasing more than the quantity r^ℓ from the firms offering the lowest price. For the indifference curves given in Figure 1, the commodity bundle A with $r^\ell + x$ units of X is optimal given the budget constraint and the rationing constraint, and bundle B with D (p_i) units of X would be optimal if good X was only offered at the price p_i. If X is not an inferior good, then $D(p_i) \leq r^\ell + x$, and summing across all consumers under these conditions we find $D(p_i) \leq \Sigma r^\ell + D_i(s) = \sum_{j \epsilon M} q_j + D_i(s)$. Thus, we see that if the good is not an inferior good for all consumers, condition (iii) must be satisfied with any rationing mechanism.[8]

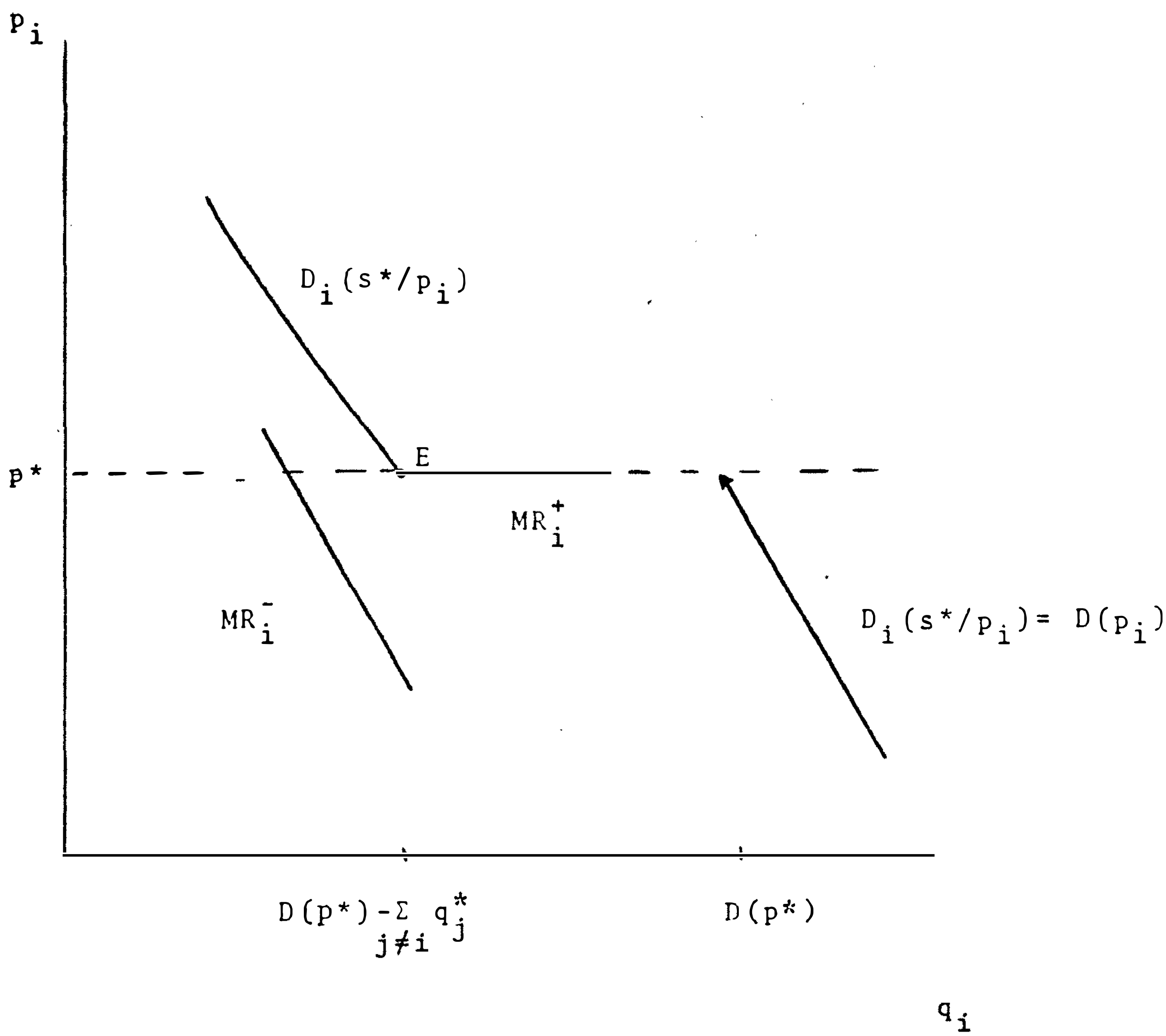
III. Equilibria in the Oligopoly Model with Homogeneous Products

Now we wish to analyze the Nash equilibria of our oligopoly game, selections of strategies s^* where

$$\pi_i(s^*) = \max_{p_i \geq 0, q_i \geq 0} \pi_i(s^*/p_i, q_i) \text{ for all } i = 1,2,\ldots,n.$$[9]

THEOREM 1: For any oligopoly ($n>1$) with individual demand functions (D_i) which are consistent with a differentiable market demand function D and with differentiable cost functions (C_i), no equilibrium exists where all firms are earning positive revenues.

In the proof of this theorem[10] we first demonstrate that in an equilibrium all firms charge the same price and markets clear, a result common to those models which require market clearance, but here it is deduced from the model rather than being one of its initial assumptions. The group of firms offering the lowest price won't undersupply the market, as otherwise each firm would have an incentive to increase its price by at least a small amount and leave its quantity unchanged. If these firms do not undersupply the market, no demand remains for any other firms, so that if all firms are to earn positive revenues they all must charge the same price. These firms won't oversupply the market either, as some firm would have an incentive to decrease its price, where it has all of the market demand instead of a share of it, which leads to a relatively large change in the amount sold and virtually no change in price. We then demonstrate that in an equilibrium each firm attains a profit level achievable by a local maximum if the firm acted as a price-taker, a result consistent with the spirit of the Bertrand solution. This result must hold as any firm can utilize all of the market demand with an arbitrarily small drop in price. The third part of the proof demonstrates that in an equilibrium no firm can increase its profit by increasing its price and acting



$$R_i(q_i) \equiv \sup_{p_i} \; p_i q_i \quad \text{subject to} \quad q_i \leq D_i(s^*/p_i)$$

$$MR_i^-(q_i) \equiv \lim_{\substack{\varepsilon \to 0 \\ \varepsilon < 0}} \frac{R_i(q_i+\varepsilon) - R_i(q_i)}{\varepsilon}$$

$$MR_i^+(q_i) \equiv \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{R_i(q_i+\varepsilon) - R_i(q_i)}{\varepsilon}$$

Figure 2: The Decision for Firm i.

such a model, and given the choices of the more complete model, we find Cournot behavior is irrational, in that some firm has an incentive to decrease its price and increase its quantity from the choices made in the Cournot outcome.[12] The perfectly competitive model, yielding the outcome suggested by Bertrand, is also not such a model as we find perfectly competitive behavior is irrational for many typical markets, in that each firm has an incentive to increase its price and decrease its quantity from those choices made in the perfectly competitive outcome. We should note that in the Cournot mineral springs example, which is all that Bertrand discussed, a perfectly competitive outcome is an equilibrium outcome in this model. However, after imposing any positive marginal costs in this example, we find no equilibrium exists, so that Bertrand's result is somewhat anomalous.[13] Edgeworth's result, on the other hand, occurs quite frequently and this theorem can be viewed as a generalization of Edgeworth's result, along with a more rigorous development of this result which was lacking in the original work.

Formally, consider a model structured as a game where n firms are the players, a strategy for each firm i is a choice of a location ℓ_i, a nonnegative price p_i, and a nonnegative quantity q_i once each firm is given full information, and each firm's payoff is its profit. To have full information each firm must know all individual demand functions (D_i) which are consistent with each individual consumer's market demand function D, and it must know all cost functions (C_i). Then where $s \equiv (\ell_1, p_1, q_1, \ldots, \ell_n, p_n, q_n)$ the payoff to firm i is

$\pi_i(s) \equiv p_i \min \{D_i(s), q_i\} - C_i(q_i)$.

The individual demand functions (D_i) are defined to be consistent with each individual's market demand function D if

for every s, $D_i(s) = \int_{-\infty}^{\infty} D_i^{\ell}(s)dF$ where D_i^{ℓ} is the demand for

firm i from the consumer at location ℓ, where (D_i^{ℓ}) are consistent with D, and F is the distribution of consumers over the real line. Let the set of locations where the delivered cost of the good from firm i is lowest be denoted by $L_i(s)$, i.e., $\{\ell : p_i + c(|\ell_i - \ell|) < p_j + c(|\ell_j - \ell|)$ for all $j \neq i\}$ where c is the transport cost function. We define (D_i^{ℓ}) to be consistent with D if for every s where some firms choose the same location the consistency conditions in Section II are met and for every s where all firms choose different locations, $D_i^{\ell}(s)$ is nonnegative, $D_i^{\ell}(s) = D(p_i + c(|\ell_i - \ell|))$ for all $\ell \in L_i(s)$, and $D_i^{\ell}(s) = 0$ if with an atomless distribution of consumers F, no equilibrium exists where some market areas overlap and all firms are earning positive revenues.

The proof of this theorem is similar to the proof of Theorem 1. The first part of the proof demonstrates that all markets clear, so that each consumer purchases all he wants from his favored supplier and each firm sells all it produces. Again market clearance is a result derived from the model, not one that is assumed a priori. We show each firm i will not undersupply the consumers in $L_i(s^*)$, as in this case the firm would have an incentive to increase its price and leave its quantity unchanged. If no firms undersupply these customers, no demand for an individual firm i will come from consumers not in $L_i(s^*)$. Firms won't oversupply the market either, as a strictly increasing cost function would give each firm an incentive to decrease its quantity to the amount that is actually sold and leave the price unchanged. The second part of the proof demonstrates that in an equilibrium each firm attains a profit level achievable by a local maximum if the firm acted as a price-taker. This result follows because each firm has the option of choosing the same location as another firm whose market area overlaps its own and undercutting its price slightly, so that the firm now has all of the market clearance, that does produce the same equilibria. If a model is constructed where firms choose their location and their price, and the "environment" determines to produce for each firm i the amount demanded by those consumers in $L_i(s/P_i)$, then the same equilibria result. In this case we find including an unconstrained quantity choice for each firm is not essential, while having unconstrained location and price choices is essential. Given this, we note that any location model which uses the standard assumptions and does not allow unconstrained location and price choices yields predicted outcomes which are irrational in the more complete model.[18]

V. Summary

Using a game theoretic structure, we have constructed some oligopoly models in which only standard assumptions are made, except that we allow the decisionmakers in the model the same choices available to the actual decisionmaker being modeled. The standard assumptions include: the use of the static framework, the assumption of full knowledge of the market and its participants by each of its participants, the assumed effects of buyer behavior and input markets, and the use of the Nash equilibrium concept. The only unusual feature is that in the homogeneous goods case each firm is given explicit control over both the price it charges for its goods and

In particular, this model seems to allow the evaluation of many other traditional oligopoly models such as the predatory pricing models, the limit pricing models, and the reaction function models. Also, many features of the oligopolistic environment which have seemed important to those doing empirical work, such as durability and inventories, bankruptcy and wealth, and any costs of adjusting the physical actions from one period to the next,[20] seem to have an effect in the dynamic model, whereas in the static model they do not. Of course, this richness is not achieved without cost as this framework is much more complex than the usual static framework. Much more work needs to be done with this dynamic model.

The second assumption which seems to have a drastic effect on predicted behavior if relaxed is the information assumption. One possibility is to change the information concerning other sellers which is either revealed by the market as a by-product of its operation or communicated by other sellers. Another possibility is to relax the assumption that all buyers receive all market information costlessly, so that there may be a search process or advertising to spread this information. Either change seems to have an effect on the predicted outcomes, especially if the dynamic framework is used.

A major lesson from this work is that any complete model of decisionmaking behavior must describe exactly the

APPENDIX

Proof of Theorem 1

Assume $s^* \equiv (p_1^*, q_1^*, \ldots, p_n^*, q_n^*)$ is an equilibrium where all firms are earning positive revenues, i.e., $p_i^* \min\{D_i(s^*), q_i^*\} > 0$ for all i. Also, assume (D_i) are consistent with a differentiable D and (C_i) are differentiable. Let $p^* \equiv \min\{p_i^*\}$ and $M \equiv \{i : p_i^* = p^*\}$.

(i) Show $p_i^* = p^*$ for all i and $\sum_i q_i^* = D(p^*)$.

(a) If $\sum_{i \varepsilon M} q_i^* < D(p^*)$, then for all $i, q_i^* < D_i(s^*)$.[22] In this case, there exists an $\varepsilon >$ such that

$$\begin{aligned}
\pi_i(s^*/p_i = p^* + \varepsilon) &= (p^* + \varepsilon) \min\{D_i(s^*/p^* + \varepsilon), q_i^*\} - C_i(q_i^*) \\
&\geq (p^* + \varepsilon) \min\{D(p^* + \varepsilon) - \sum_{\substack{j \varepsilon M \\ j \neq i}} q_j^*, q_i^*\} - C_i(q_i^*) \\
&= (p^* + \varepsilon)\, q_i^* - C_i(q_i^*) \\
&> p^* q_i^* - C_i(q_i^*) \qquad = \quad \pi_i(s^*),
\end{aligned}$$

contradicting the assumption that s^* is an equilibrium. Thus, at such an equilibrium $\sum_{i \varepsilon M} q_i^* \geq D(p^*)$.

(b) If $\sum_{i \varepsilon M} q_i^* \geq D(p^*)$, then $D_j(s^*) = 0$ for all $j \notin M$.

For all firms to have positive revenues, it must be that $M = \{1, 2, \ldots, n\}$ so that $p_i^* = p^*$ for all i.

(c) If $\sum_{i \varepsilon M} q_i^* > D(p^*)$, then for some $i, q_i^* > D_i(s^*)$.[23]

incentive for any firm i to increase price with any feasible quantity at an equilibrium s^*,

$$\pi_i(s^*) \geq \sup_{p_i, q_i} \pi^i(s^*/p_i, q_i) \text{ subject to}$$

$$p_i > p^*, q_i = D(p_i) - \sum_{j \neq i} q_j^* \geq 0$$

$$= \sup_{p_i} p_i(D(p_i) - \sum_{j \neq i} q_j^*) - C_i(D(p_i) - \sum_{j \neq i} q_j^*)$$

$$\text{subject to } p_i > p^*,\ D(p_i) - \sum_{j \neq i} q_j^* \geq 0$$

$$\geq p^*(D(p^*) - \sum_{j \neq i} q_j^*) - C_i(D(p^*) - \sum_{j \neq i} q_j^*) = \pi_i(s^*).$$

This implies $\pi_i(s^*) = \max_{p_i} p_i(D(p_i) - \sum_{j \neq i} q_j^*) - C_i(D(p_i) - \sum_{j \neq i} q_j^*)$

subject to $p_i > p^*$ and $D(p_i) - \sum_{j \neq i} q_j^* \geq 0$ for all i.

(iv) Show that the results in parts (ii) and (iii) lead to a contradiction. From (ii), $\pi_i(s^*) = \max_{0 < q_i < \varepsilon} p^* q_i - C_i(q_i)$,

which implies $p^* - \frac{dC_i}{dq_i}(q_i^*) = 0$. From (iii),

$$\pi_i(s^*) = \max_{p_i} p_i(D(p_i) - \sum_{j \neq i} q_j^*) - C_i(D(p_i) - \sum_{j \neq i} q_j^*) \text{ subject to}$$

$p_i > p^*$ and $D(p_i) - \sum_{j \neq i} q_j^* \geq 0$, which implies

Thus, in such an equilibrium $q_i^*=D_i(s^*)=\int_{L_i(s^*)}D_i^{\ell}(s^*)dF$ for all i.

(ii) Consider firms i and j whose market areas overlap.

Show that $\pi_i(s^*) = \max_{0<q_i<\varepsilon} p_i^* q_i - C(q_i)$ for some $\varepsilon > 0$

Since s^* is an equilibrium,

$$\pi_i(s^*) \geq \sup_{p_i<p_j^*,q_i} \pi_i(s^*/\ell_j^*, p_i, q_i)$$

$$\geq \sup_{q_i} p_j^* \min\{\int_{L_j(s^*/p_i=\infty)} D_j^{\ell}(s^*/p_i=\infty)dF, q_i\} - C(q_i)$$

$$\geq p_j^* \min\{D_j(s^*), q_j^*\} - C(q_j^*) = \pi_j(s^*).$$

Similarly, $\pi_j(s^*) \geq \pi_i(s^*)$, so that all of the inequalities must be equalities. This means

$$\pi_i(s^*) = \max_{q_i} p_i^* \min\{\int_{L_i(s^*/p_j=\infty)} D_i^{\ell}(s^*/p_j=\infty)dF, q_i\} - C(q_i)$$

and since the market areas overlap,

$\int_{L_i(s^*/p_j=\infty)} D_i^{\ell}(s^*/p_j=\infty)dF > D_i(s^*) = q_i^*$, so that

$$\pi_i(s^*) = \max_{0<q_i<\varepsilon} p_i^* q_i - C(q_i) \text{ for } \varepsilon = \int_{L_i(s^*/p_j=\infty)} D_i^{\ell}(s^*/p_i=\infty)dF.$$

FOOTNOTES

1. Examples of recent work using the Cournot model are: Ruffin (1971), Kamien and Schwartz (1975), Gabszewicz and Vial (1972), Roberts and Sonnenschein (1977), Novshek (1977), and Novshek and Sonnenschein (1978).

2. e.g. Friedman (1977), p. 39.

3. An example of such a complete information game is the following reformulation of the Walrasian model: The players in the game are consumers, firms, and a Walrasian auctioneer. A consumer can choose any commodity bundle within his budget constraint, a firm can choose any bundle of feasible inputs and outputs, and the auctioneer can choose any prices. The payoffs for the consumers and firms are the utilities received with the chosen bundles, and for the auctioneer payoffs are maximized if markets clear. Note that these payoffs are attainable for the consumers and firms if and only if the quantities people choose to buy equal the quantities people choose to sell.

4. This idea of demand for the individual firms is not new. They would be identical to the "contingent demand" functions in Shubik (1959), if we assumed the individual demand for each firm is independent of its own quantity. We don't make this extra assumption because it is not needed for later results.

5. Requiring $D_i(s) \geq D(p) - \sum_{\substack{j \in M \\ j \neq i}} \min \{D_j(s), q_j\}$ for all $i \in M$

   and $\sum_{i \in M} \min \{D_i(s), q_i\} \leq D(p)$ is essentially equivalent to

   condition (i), which means demand functions satisfy the conditions above if and only if other demand functions satisfy condition (i), where both yield the same physical outcome.

6. If $\sum_{i \in M} q_i \leq D(p_i)$ and $q_i > D_i(s)$ for some $i \in M$, then

   $D(p) = \sum_{\substack{j \in M \\ j \neq i}} \min \{D_j(s), q_j\} + D_i(s) < \sum_{i \in M} q_i$, yielding a

   contradiction.

12. Exceptions may arise when, for example, the Cournot outcome is also a perfectly competitive outcome, but the markets that allow these exceptions are uninteresting. See Alger (1979), pp. 37-45.

13. See Alger (1979), pp. 37-45.

14. "I opt for the Cournot version because... (when) differentiated products price models are taken up, it is seen that the insight gained from such models is much more nearly approached by the Cournot than the Bertrand version." p. 39, Friedman (1977).

15. Only the last condition may not be obvious. Note that when all firms choose different locations ($L_i(s)$) partitions the real line except for a finite number of points where some delivered costs are equal. When $q_j \geq \int_{L_{j(s)}} D_j^{\ell}(s)dF$ for all j, each consumer can purchase all he wants from the supplier giving him the lowest delivered cost so that there is no additional demand for any other supplier.

16. Other work that shows equilibria may not exist include: Salop (1976) and D'Aspremont, Gabszewicz, and Thisse (1979).

17. As an example, in Salop (1976) it is shown that no equilibria exist where markets overlap (pp. 46-47), and the result is ignored.

18. Some work that uses this type of location model includes: Eaton and Lipsey (1975) and Hart (1979).

19. See Chapter V in Alger (1979).

20. In the models presented we have considered using only pure strategies. If mixed strategies are considered, as well, equilibria may exist, where only strictly mixed strategies could be used, as indicated by similar work in Shilony (1977) and Varian (1978). However, if there are costs of adjustments these strictly mixed strategies are likely to be eliminated, so as not to pay these costs in every period.

21. Of course, to adequately describe the set of feasible choices for each decisionmaker, some attention needs to be paid to the actual economic institution being used. In the oligopoly model presented here, we specifically modeled behavior under the posted offer institution, but I expect that equilibrium behavior is not changed with any institu-

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