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SPATIAL COMPETITION WITHIN AN OPTIMAL CONTROL FRAMEWORK

The widespread argument that competition among spatially separated firms differs significantly from that among neighboring firms has been convincingly upheld. Recently, there has been a number of articles, both theoretical and empirical, examining the pricing policies of these spatial competitive firms. Greenhut and Ohta, in their book on market areas, demonstrated when discriminatory pricing would be profitable for a spatial competitive firm. In an entirely different model, Greenhut and Greenhut went further in demonstrating the effects of competition on spatial prices. Finally, in a very recent article, Norman showed that spatial competition can lead to an increase in the degree of price discrimination.

In this paper we will continue the examination of the effects of spatial competition on pricing. By generalizing the model of the spatial firm to derive the competitive prices, we will add to the understanding of how these spatial competitive firms operate. To be more specific, the model of the spatial firm will be generalized in two ways. Most importantly, the model will include a general production cost function. Previous articles on this topic have included models with production cost functions characterized by constant marginal production cost. The model is also generalized by not restricting the shape of the firm's delivered price schedule.[1] This change in the model will have a significant effect on the solution derived.

To accomplish these generalizations we will use an optimal control framework. This maximization technique allows us to derive the firm's entire delivered price function at one time, a requirement caused by the general production cost function.[2]

We begin the paper by reviewing the literature on spatial competitive pricing. That review deals with the important spatial competitive pricing research of Greenhut and Ohta, as well as the very recent research of Norman. A careful description of this literature will make it easier later to identify the significance of this paper's new results.

Following the literature review, we will explain the optimal control framework and the derivation of the schedule of delivered prices. After that presentation, we will compare the derived solution to results that others have found. We will then examine each of the significant reasons why our results are different. In the final section of the paper we will discuss our results and suggest a further extension of this model.

I. PREVIOUS SPATIAL COMPETITIVE PRICE LITERATURE

A. Greenhut and Ohta--Competitive Price Results

An interesting model of spatial competition was suggested by Greenhut and Ohta in their book on market areas and spatial price discrimination. Their competitive model rests on the fact that the curtailed market areas of spatial competitive firms are caused by the prices that the distantly located firms charge.[3] At some point between the two firms, the two delivered-price functions intersect and the delivered prices that each firm charges are the same. The market area of each of the two firms will exist on one side of that point. Consequently, at that market area boundary the firm faces a maximum-price constraint imposed by its competitor. Greenhut and Ohta state,

> . . . the firm visualizes a given maximum delivered price when a distant rival competes with it for selected buyers. [p. 129]

A firm assuming a maximum price constraint at its boundary is referred to as a Greenhut-Ohta (GO) firm.

With this competitive model, Greenhut and Ohta were the first to examine how the entry of distantly located firms would affect both the delivered prices charged and the market areas selected by the firms faced with distantly located competition. To be more specific, these authors examined how discriminatory-pricing firms[4] react when new firms have entered away from the existing firm.

Before discussing their results, we should mention that the Greenhut-Ohta analysis of spatial competition included one important feature concerning how the firm determines its delivered price function. The spatial firm will determine the discriminatory delivered price at each point by maximizing the profits obtained at that point. The delivered price charged at every point will be independent of every other point's delivered price.[5] Consequently, after new firms enter, the discriminatory prices a firm charges at the points still in its market area are the same as those it charged before the new firms entered. Thus, the discriminatory firm will not change the degree of discrimination (i.e., how prices change with distance from the firm) when the level of competition changes.

As will be discussed below, Norman thought that this response was too restrictive, and that in the presence of increased competition, firms should be allowed to change the level of discrimination. However, with the GO model, the constraint on the delivered price at the boundary point produces a workable method of incorporating spatial competition. Since the delivered prices will always increase with distance from the firm,[6] the only point where the maximum price (imposed by the new distantly located rival) can be reached is at the boundary.

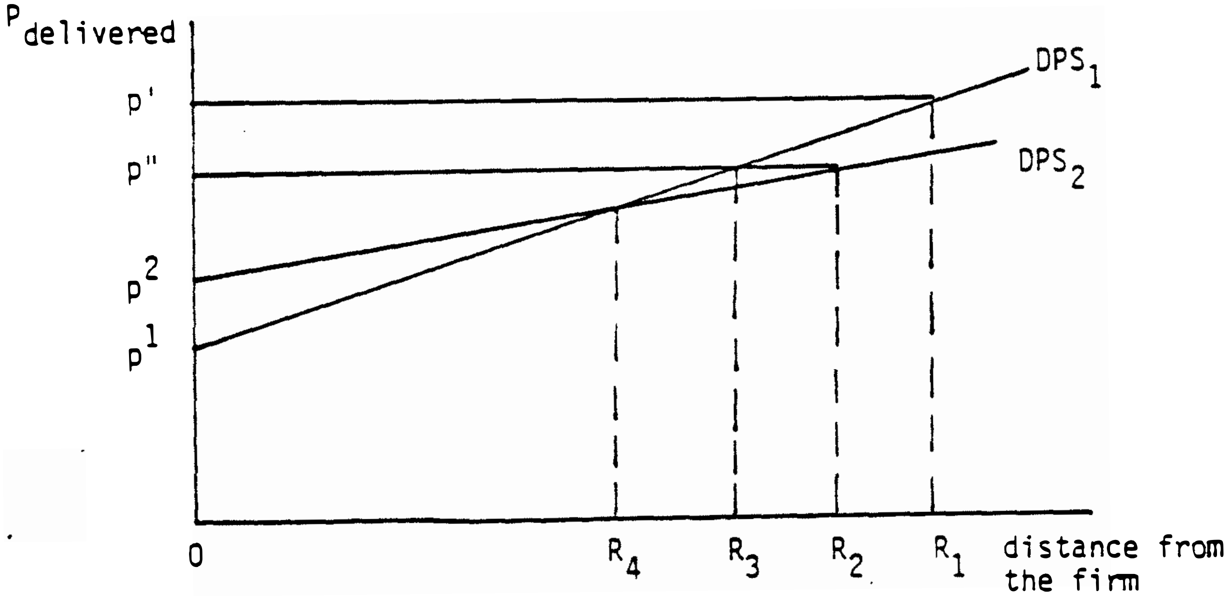
Perhaps the most significant result these authors found using this model is that after significant competition occurs, a firm will find nondiscriminatory pricing more profitable than discriminatory pricing. Entry of new, distant firms constricts the market area of an existing firm. If there is enough entry and if the market areas are small, the prices that a discriminating firm charges are higher at every point in the ensuing market area than those of a nondiscriminating firm. Also, the market area of a nondiscriminating firm is larger. In that situation, a nondiscriminating firm will sell a greater quantity and earn larger profits than a corresponding discriminating firm would.[7] Greenhut and Ohta have found in their model, therefore, that if spatial competition is carried far enough, nondiscriminatory pricing will displace discriminatory pricing.

B. Norman--Competitive Price Results

A recent paper by Norman adds to the spatial-competitive-price literature by proving that both spatial price discrimination and spatial competition must occur simultaneously in a GO competitive model. Under Norman's assumptions, a GO competitive firm will try to offset the negative impact of increased competition by increasing the degree of discrimination. Moreover, in Norman's model, if competition progresses far enough it will become most profitable for a firm to price uniformly in its own market area.

We will use a simple diagram to explain Norman's model and results. The spatial firm in figure 1 initially faces a competitive price constraint of p' at its market boundary and selects the delivered price schedule DPS_1. A new distantly located firm then enters, driving the price constraint down to p''. If the older firm keeps the same price schedule in its new market area, its market area size becomes OR_3. If instead it changes the delivered price schedule--say, to DPS_2--the market area size becomes OR_2. The firm loses less of its original market area by increasing the degree of discrimination (i.e., by lowering the amount by which prices increase with distance from the firm). In the diagram below, the firm raises the delivered prices to customers located closer than distance OR_4 so as to expand its market area to distance OR_2.

Figure 1.



$P_{delivered}$
p'
p"
p^2
p^1
DPS_1
DPS_2
0
R_4
R_3
R_2
R_1
distance from the firm

To explain why Norman's price result is different from the corresponding GO result, we must describe Norman's method of deriving the firm's delivered price schedule. Unlike Greenhut and Ohta, Norman derived his delivered price schedule by maximizing the profits obtained from all points in the market area simultaneously. By choosing the price at the plant site and the change in prices within the firm's market area (i.e., the slope of the price schedule), the managers of the spatial firm in Norman's model selected the entire delivered price function at once. By contrast, Greenhut and Ohta allowed the manager to maximize separately the profits obtained from each point, in order to derive first the delivered price charged at that point, and ultimately the entire delivered price function. When determining the entire delivered price schedule, the manager in Norman's model could sacrifice profits obtained from closer points, in order to retain some distant customers the firm otherwise would have lost. This reaction will maximize the total profits the firm can obtain in Norman's model.

In addition, there is a significant restriction in Norman's model. He only allows straight-line delivered price schedules to be selected by the firm. The restriction is significant because the only way a manager extends his market area is by "tilting" the entire delivered price schedule, i.e., increasing the degree of discrimination. Once this restriction is removed (as we will see later in the paper), the manager can reach distant customers without "tilting" the entire price schedule; consequently, he will choose a different schedule.

Finally, Norman also found that sufficiently forceful competition would lead to uniform pricing.[8] After the competitive-price constraint is driven low enough (the level depending on cost and demand conditions), the manager of the spatial firm will find it profitable to charge that price throughout his market area. This paper will show that in the general model this result will not occur as frequently as Norman suggested, if at all.

II. OPTIMAL CONTROL FRAMEWORK

Since optimal control theory allows the analyst to examine situations that exhibit change in many, if not all, of the relevant variables, the theory is well suited for examining the nature and effects of spatial competition. Within a firm's market area, a number of relevant variables could be changing. For example, transportation costs, customer density, and--most important--the delivered price, could all be changing as distance from the firm increases.

The manager of the spatial firm will acknowledge these changing variables when he selects a delivered price schedule to maximize the total profits of the firm. Before we can express the total profits that will be maximized under our framework, we must mention the few simplifying assumptions of our model. First, we assume that all customers are distributed uniformly, along a line (perhaps along some interstate highway), each with the same demand schedule, and that the quantity demanded at any point depends on the delivered price at that market point. Second, we arbitrarily set the extent of the firm's market--its market area size. Obviously this is a variable that the manager of the firm chooses, but for our purposes it can be taken as given. Third, the firm is subject to two costs--transportation and production. The unit transportation cost to any point in the market area depends only on the distance from the firm. The production cost function depends on the total quantity sold and is not specified. Using a general production cost function here is important, since most previous work in this area assumed a production cost function characterized by constant marginal production costs. As demonstrated later, the firm's production cost function has a significant impact on the final solution we derive.

Now that we have described the position of the spatial firm, we can express the value of total profits. Total profits equal

$$\pi = \int_u^R [q(p(r))\, p(r) - t(r)q(p(r))]\, dr - C(Q(R)),$$

where $r \equiv$ distance away from the firm
$R \equiv$ market area size
$p(r) \equiv$ delivered price function
$q(p(r)) \equiv$ demand function
$t(r) \equiv$ unit transportation cost function
$Q(R) \equiv$ total quantity sold, i.e. $\int_0^R q(p(r))dr$
$C(Q(R)) \equiv$ production cost function.

Within our optimal control framework, total profits of the spatial firm are maximized by maximizing the complete contribution that each point makes. The complete contribution consists of two parts--the direct effect of the revenues obtained and transportation costs incurred from the quantity sold at each point, and the indirect effect of the production costs of the quantity sold at each point. The latter effect is indirect, since total production costs do not depend on the quantity sold at each point but on the quantities sold at all the points in the market area.

For this reason, we must take into account the relationship between the quantity sold at each point and the total quantity sold. In fact, total quantity is the state variable in our optimal control model, which is constrained at each point in the market area by the quantity sold at each point. In other words, the total quantity sold must increase at every point by the amount sold at that point. In equation form, this relationship is

[2] $\dot{Q}(r) = q(p(r))$

Associated with that constraint is an auxiliary variable, $\lambda(r)$, which equals the change in total profits brought about by a change in the level of that constraint, i.e., the change in the total quantity sold at that point. In other words, the auxiliary variable will be equal to the amount that total profits change because production costs have changed (brought on by the increase in the level of the constraint).

Finally, before we can derive the delivered price function, we must incorporate the impact of distant competition into the model. Following the method that Greenhut and Ohta first proposed and Norman later used, we will assume that the delivered price at every point in the market area of the firm must be lower than or equal to a maximum price p'. Obviously, a more realistic assumption would be that a firm faces a rival's entire delivered price function as a maximum-price constraint. But so that we can compare our model with previous models and simplify the analysis, we assume one maximum price throughout the market area. This constraint on the delivered price, the choice variable, can be written as

$$[3] \quad p(r) \leqslant p'.$$

This constraint also has a value at every point in the market area, $\mu(r)$. That value equals the change in the contribution to total profits that each point makes with a change in the level of the maximum-price constraint--i.e., a change in the level of competition.

Now that we have described the position of the firm, we can express the Hamiltonian, which represents the complete profits obtained at any market point r. The Hamiltonian value is

$$[4] \quad H(r) = q(p(r))p(r) - q(p(r))\, t(r) + \lambda(r)q(p(r)) - \mu(r)(p(r)-p').$$

Using the maximum principle first proved by Pontryagin et al., we can write out the five necessary conditions for obtaining the solution. These conditions are

$$[5] \quad \frac{\partial H(r)}{\partial p(r)} = q'(p(r))p(r) + q(p(r)) - q'(p(r))\, t(r) + \lambda(r)q'(p(r)) - \mu(r) = 0$$

$$[6] \quad \frac{\partial H(r)}{\partial \lambda(r)} = \dot{Q}(r) = q(p(r))$$

$$[7] \quad -\frac{\partial H(r)}{\partial Q(r)} = \dot{\lambda}(r) = 0$$

$$[8] \quad \lambda(R) = -C'(Q(R))$$

$$[9] \quad \mu(r)\,(p(r) - p') \leqslant 0.$$

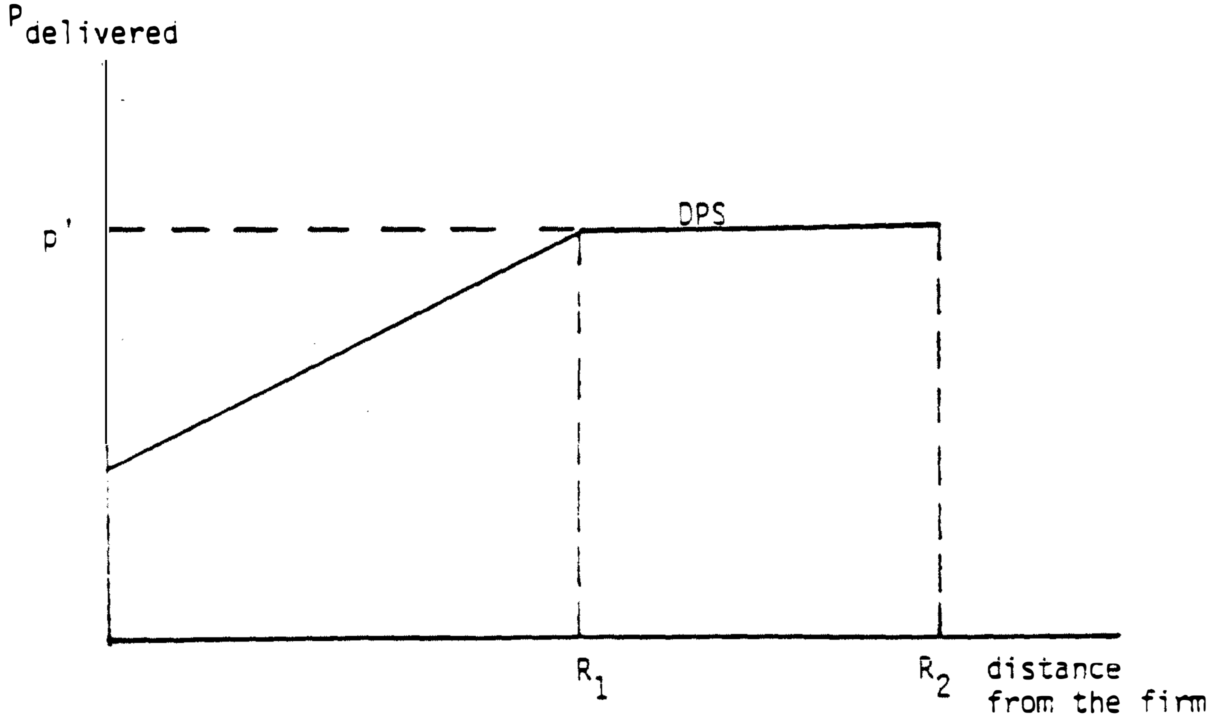
III. THE SOLUTION OF DELIVERED PRICE SCHEDULE

Using equations [5] through [9], we can express the profit-maximizing delivered price schedule. That solution is

$$[10]\quad \hat{p}(r) = -\frac{q(\hat{p}(r))}{q'(\hat{p}(r))} + t(r) + C'(Q(R)) + \frac{\mu(r)}{q'(\hat{p}(r))} .$$

At points where the constraint is not binding, $\mu(r)$ is equal to zero and the delivered price function is equal to the first three expressions on the right side of the equation. At all other points, the delivered price equals the level of the maximum price constraint and $\mu(r)$ is not zero. Figure 2 below represents one solution.

Figure 2.

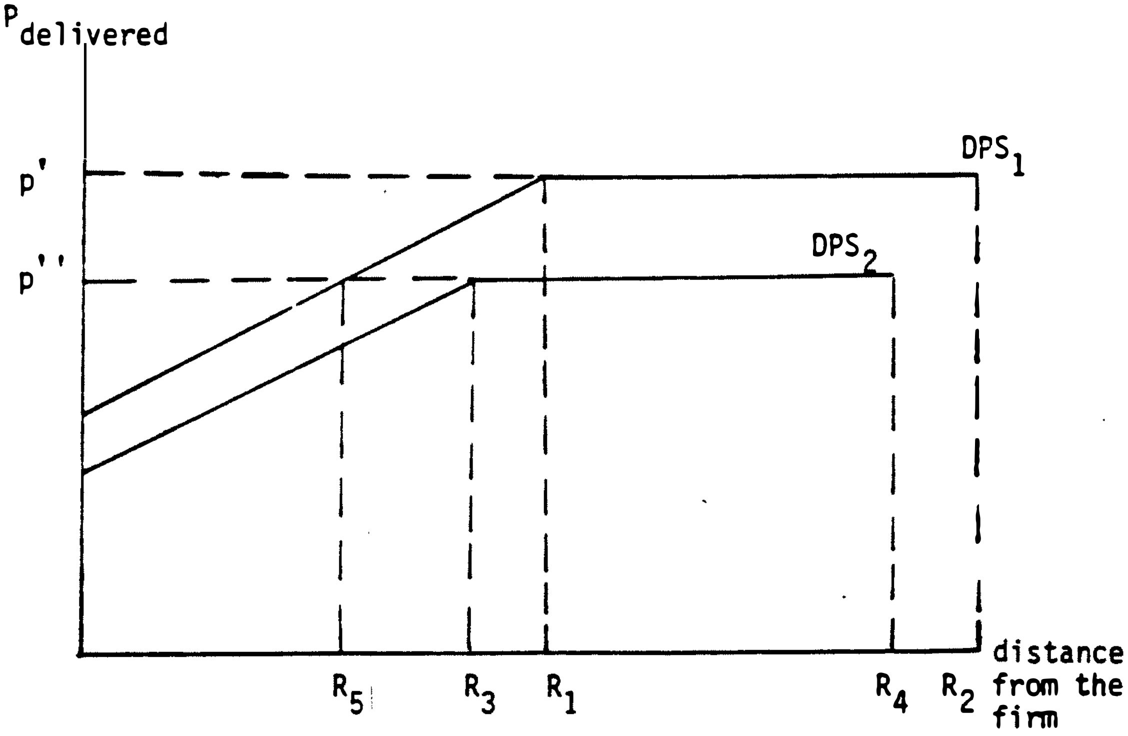


From this diagram it is obvious that the effect of competition with our generalization of the spatial firm's model differs from the effect found in other models. In fact, the delivered price function we derived is a combination of the GO and Norman results. Over some of the market area of the firm, the degree of discrimination (i.e., slope of the delivered price schedule) is the same as it would be without distant competition. In the rest of the market area, the firm follows uniform pricing. The result differs from the GO result because the profits obtained at all points are maximized at the same time. Consequently, some prices will change as competition is introduced. Likewise, one reason the result differs from Norman's is that the delivered price schedule is not constrained to be a straight line. Therefore, the entire delivered price schedule need not be "tilted" to reach distant customers.

Another reason this solution differs from those two models is the simultaneity aspect of the spatial pricing problem included in this model. The delivered price charged until the point R_1 will depend partly on the level of the marginal production cost that is determined by the total quantity sold in the entire market area. Obviously the total quantity sold depends on the delivered prices charged throughout the market area. In fact, since the upward-sloping portion of the delivered price function depends on the prices charged throughout the market area, the market point where the value of the delivered price function reaches the price constraint (in figure 2, location R_1) also depends on the prices charged throughout the market area. To summarize, the total quantity sold from the site of the mill to the market area boundary point R_2 determines both the level of the delivered prices from the mill to some point R_1 and the distance of R_1 from the firm. In turn, the prices charged up to location R_1 determine the level of the total quantity sold. This simultaneity aspect of the solution was not present in either the GO model or the Norman, since neither included a general production-cost function. In that literature, the amount sold at one point did not affect the amount sold at any other point through the impact on production costs.[9]

Furthermore, this simultaneity aspect also influences the impact of further increases in competition. An increase in competition implies that there is a lower maximum price constraining the spatial firm. As we have said, Norman under his assumptions found that because of this new lower constraint, the degree of discrimination will increase. Greenhut and Ohta under their assumptions proved that the degree of discrimination will remain constant. Under the assumptions in our model, once again a combination of these two results could occur. The firm discriminates over some of its market area by the same degree, while in the rest of the resulting market area it follows uniform price behavior.[10] Figure 3 below shows this process by suggesting a new delivered price schedule that would be possible after the increase in competition.

Figure 3.



P delivered
p'
p''
DPS1
DPS2
R5
R3
R1
R4
R2
distance from the firm

In figure 3 the firm initially faces the competitive price constraint p' and follows the delivered price schedule DPS_1. New firms enter, driving the competitive price constraint down to p''. First, assume that marginal production costs are constant. If so, the upward-sloping portion of the delivered price schedule does not change with the total quantity sold. Therefore, the delivered price schedule of the firm will still be DPS_1, but now only to the point R_5. Uniform pricing would be continued throughout the rest of the market. Now, assume that marginal production costs are increasing. In this case, as the total quantity sold by the firm decreases because of the new competition, so will the level of the marginal production costs used in determining the delivered price schedule. Consequently, the delivered price function is lowered, such as to DPS_2 in figure 3. With that schedule, uniform pricing is followed only after point R_3.[11] It is important to note, however, that in the area where uniform pricing is not followed, the degree of discrimination practiced is the same as before the increase in competition occurred.

Finally, the rate of increase (decrease) in the marginal production cost function also determines the area over which uniform pricing is followed after an increase in the level of competition. This rate of increase (decrease) will determine how much prices will decrease (increase), once new firms have entered. For example, suppose the firm faces increasing marginal production costs, but in this case those costs increase more slowly than the marginal production costs used to determine the delivered price function DPS_2 in figure 3. Then, as the competitive price constraint falls to the level of p'' and a smaller total quantity is sold, the decrease in the level of marginal production costs used to determine the delivered prices is less. As a result, those prices will not decrease as much as the delivered prices of DPS_2. Furthermore, the location where the price constraint becomes binding, i.e., where uniform pricing is followed, falls to the left of location R_3.

IV. CONCLUSION

We have demonstrated that generalizing the model of a spatial firm significantly affects competitive price results. This change in results, however, should not come as a surprise, since we have significantly modified the model previously used. What should be surprising is how easily those modifications have been made. In other words, the optimal control framework has been shown to be an extremely applicable method of generalizing the model of the spatial firm and analyzing such firms.

In fact, we have been able to clear up a recent controversy in the spatial-competitive-price literature with the general model presented. By allowing the firm to choose an entire unconstrained delivered price schedule at one time, we have been able to show that a combination of the previously found results will emerge, a firm could price uniformly only over some of its market area. Furthermore, we have also been able to demonstrate that the effects of an increase in competition will depend, in part, on the production cost function facing the spatial firm.

Finally, it appears that additions or complications to the general model presented here can easily be incorporated. For example, as we mentioned before, the spatially competitive firm is not confronted with one competitively imposed maximum price. Instead, the various delivered prices that the firm's competitor charges at each point must be either matched or undercut in order for the firm to sell at that point. To include that complication in the optimal control model, a variable constraint (namely, the competitor's delivered price function) would be imposed on the firm's choice variable. A new delivered price function could then be derived.

FOOTNOTES

[1] A minor restriction that the delivered prices must increase by less than the increase in transportation costs is placed on the model. This restriction is minor, since the majority of demand curves will generate this outcome anyway.

[2] It is interesting to note that in a recent article Spulber also employs an optimal control framework to derive the delivered price schedule of a monopolist offering quantity discounts and facing constant marginal production cost.

[3] We are only examining the cases in which the spatial firms have their own distinct market areas. In other words, there is no area in which both firms sell.

[4] Discriminatory prices within spatial contexts are defined as prices charged at different points that differ by more or less than the differences in transportation costs to those points.

[5] This analysis also is contingent on the constant-marginal-production-cost assumption that was imposed. This issue will be discussed more extensively when the general model is presented (later in the paper).

[6] By maximizing profits obtained at each point individually, Greenhut and Ohta derived upward-sloping delivered price functions.

[7] For a proof of this result, see Greenhut and Ohta (pp. 133-39).

[8] In a totally different context, Greenhut and Greenhut obtained an identical result. After examining the assumptions imposed in both models, it is not surprising to see the same result appearing in both models. For a more detailed comparison, see Fratrik (pp. 101-2).

[9] In Norman's article, there was mention of a simultaneity aspect in this solution, since he derived the entire delivered price schedule at one time. Yet the production cost function played no part in that simultaneity aspect.

FOOTNOTES Continued

[10] Greenhut Ohta, and Greenhut found a result similar-to this when they examined the prices charged by a firm that divides its market into submarkets.

[11] It is important to notice that with decreasing marginal production costs, an increase in the level of competition will lead to more of the market area being charged a uniform price than in the example presented.

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